Think It Over

This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

Ants in pursuit

Awani Kumar
B-4, Forest Colony,
Vibhuti Khand, Gomti Nagar
Lucknow 226 010, India.

We consider in this article a pursuit problem involving two ants A and B moving in the coordinate plane; at time $t = 0$, let A be at the origin $(0, 0)$ and B at (l, h) where l, h are positive. Assume that B moves at a constant speed u in the negative x-direction, while A moves at a constant speed $v > u$, and always *directly towards* A (so its direction of movement is constantly changing). Can we be sure that A will meet B at some time $T < \infty$? If yes, can the catch-up time T be expressed in terms of u, v, l and h? What happens when $v = u$? Will A be able to catch up with B in this case?

Analysis

At time t, ant B has reached the point $(l - ut, h)$. Let A be at (x, y) at this point in time, and let θ be the angle which A's direction of motion makes with the positive x-axis at this point (this means that $\tan\theta = dy/dt \div dx/dt = dy/dx$); clearly, x, y and θ all vary with t. Let T be the time at which the ants meet. (We can hope that $T < \infty$.)

Considering the total motion in the x-direction, we get

$$\int_0^T (u + v\cos\theta)\, dt = l. \qquad (1)$$

Since the direction of motion of A is always directly towards B, their relative velocity of approach at time t is $(v + u\cos\theta)$. At time 0 the ants are separated by a distance $\sqrt{l^2 + h^2}$, and this distance is covered in time T; therefore

$$\int_0^T (v + u\cos\theta)\,dt = \sqrt{l^2 + h^2}. \tag{2}$$

Equations (1) and (2) yield, since u and v are constants,

$$\int_0^T \cos\theta\,dt = \frac{l - uT}{v}, \qquad \int_0^T \cos\theta\,dt = \frac{\sqrt{l^2 + h^2} - vT}{u}.$$

Therefore we have:

$$\frac{l - uT}{v} = \frac{\sqrt{l^2 + h^2} - vT}{u}.$$

The equation is easily solved to give

$$T = \frac{v\sqrt{l^2 + h^2} - ul}{v^2 - u^2}. \tag{3}$$

Observe that $T < \infty$ if $v > u$. The ants do meet after all.

For $v = u$ the equation yields $T = \infty$; the ants never meet. This seems paradoxical at first sight, but makes sense on reflection. When the speeds are equal, we find that the path taken by A is tangent to B's straight line path, but the point of tangency is at infinity! That is, B's path is an *asymptote* to A's path. For $t \gg 1$ we find ant A trailing behind ant B by an almost fixed distance and on the same straight line path (well, almost). As they are travelling at the same speed, the distance between them stays almost fixed.

It is worth examining the progress of the ants on a graph, particularly the case when $v = u$. At time t we have:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v^2, \quad \frac{dy/dt}{dx/dt} = \tan\theta = \frac{h - y}{l - ut - x}.$$

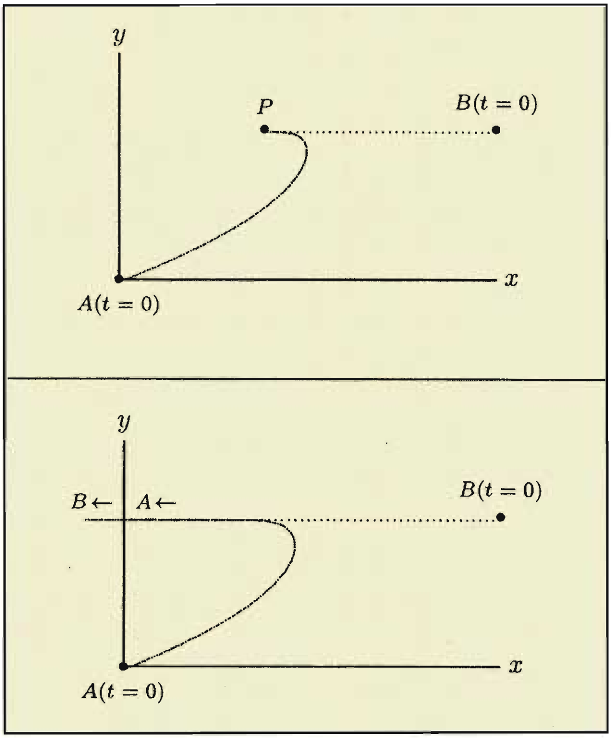

Figure 1 (top). v>u: A catches B.

Figure 2 (bottom). v=u: A never catches B.

Solving the equations for dx/dt and dy/dt we get

$$\frac{dx}{dt} = \frac{v(l - ut - x)}{\sqrt{(l - ut - x)^2 + (h - y)^2}},$$

$$\frac{dy}{dt} = \frac{v(h - y)}{\sqrt{(l - ut - x)^2 + (h - y)^2}}.$$

Solving this very intimidating pair of differential equations in closed-form would appear difficult, so we opt for an approximate graphical solution. The graphs sketched in *Figures* 1 and 2 show the paths followed by the two ants when $v > u$ and when $v = u$. In the former case the two ants meet at P, and in the latter case the ants never meet.