Minimizing the Time of Travel for a Long-distance Train Journey: A Model

S N Maitra
Department of Mathematics
National Defence Academy
Khadakwasla
Pune 411 023, India.

The problem considered here is that of optimal placement of $n-1$ railway stations $A_2, A_3, \quad, A_n$ between two given stations A_1 and A_{n+1}. A train journeys between the two end-points and is required to halt at each intermediate station. On segment $A_i A_{i+1}$ it starts from rest at a constant acceleration f_i, attains some maximum speed, keeps this speed for a while, then slows down at a constant deceleration f_i' till it stops. In an earlier article, ([1]), it was shown that for minimum time of travel, the train should accelerate to some maximum speed and then start decelerating immediately.

Let the distance between A_i and A_{i+1} be s_i. Writing t_i for the time taken for travelling from A_i to A_{i+1}, v_i for the maximum speed attained on this segment, and α_i for the quantity $1/f_i + 1/f_i'$ (which is found to repeatedly enter the analysis), we have:

$$t_i = v_i \alpha_i, \quad s_i = \frac{{v_i}^2}{2}\alpha_i. \qquad (1)$$

The problem we consider is that of choosing the placement of the $n-1$ intermediate stations so that the total time of travel for the journey is minimized.

Eliminating v_i from the two equations in (1) we get $t_i = \sqrt{2 s_i \alpha_i}$. The total time of travel is therefore T where

$$T = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \sqrt{2 s_i \alpha_i}, \qquad (2)$$

and the total distance travelled is $S = \sum_{i=1}^{n} s_i$. The total distance is of course a given constant. We must minimize T with respect to the s_i. We shall solve the problem using Lagrange multipliers. Let $F = F(s_1, s_2, \quad, s_n)$ be the function given by $F = T + \lambda\,(S - \sum_{i=1}^{n} s_i)$, i.e.,

$$F = \sum_{i=1}^{n} \sqrt{2 s_i \alpha_i} + \lambda \left(S - \sum_{i=1}^{n} s_i \right) \qquad (3)$$

Differentiating F with respect to s_i we get

$$\frac{1}{\sqrt{2}}\sqrt{\frac{\alpha_i}{s_i}} - \lambda = 0 \qquad (i = 1, 2, . \quad , n).$$

It follows that for minimum time of travel we must have

$$s_i = \frac{\alpha_i}{2\lambda^2}. \tag{4}$$

From this we get

$$\frac{s_1}{\alpha_1} = \frac{s_2}{\alpha_2} = \qquad = \frac{s_1 + s_2 + \qquad}{\alpha_1 + \alpha_2 + \qquad} = \frac{S}{\sum_i \alpha_i}.$$

Therefore the optimum value of s_i is s_i^* where

$$s_i^* = \frac{S\alpha_i}{\sum_i \alpha_i}. \tag{5}$$

From equations (1) and (5) we get for the optimum maximum velocity between stations A_i and A_{i+1};

$$v_i^* = \sqrt{\frac{2S}{\sum_i \alpha_i}}. \tag{6}$$

This equation reveals that the train attains the same maximum velocity on each segment of the journey. Hence, the minimum time of travel is

$$\sqrt{2S \sum \alpha_i}.$$

Suggested Reading

[1] S N Maitra, *Minimum time of travel, Resonance*, Vol. 7, pp. 80-83, 2002.

It is not the possession of truth, but the success which attends the seeking after it, that enriches the seeker and brings happiness to him.

Max Planck (1858-1947)

German Physicist, Nobel Prize for Physics, 1918