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Point Set Topological Proof of 'no-retraction' Theorem for 2 and 3 Dimensional Cases

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We will give the proof of the no-retraction theorem for 2 and 3 dimensional cases, using only elementary concepts from point-set topology.

Introduction

The no-retraction theorem is a well-known and very important theorem in topology. In fact the Brouwer fixed point theorem, no-retraction theorem and the essential mapping theorem are equivalent. There are simple proofs of these using homotopy or homology. There is also a proof using calculus. But we would like to give a new way of looking at this problem and we will prove it using point-set topology.

No-retraction Theorem

We will first prove the no-retraction theorem for 2-dimension case. Using a lemma we will prove it for 3-dimension case.

So let us now start by proving a lemma.

Keywords
Point-set topology, no-retraction theorem, topological manifold.

Lemma 1 *Let* $B_1, B_2, ..., B_n$ *be distinct open balls in* $\mathbb{R}^2$ *such that* $B_i \not\subset B_j \ \forall\ 0 < i, j \leq n,\ i \neq j$.

Let $\mathcal{C} = \bigcup_{i=1}^{n} B_i$. Let $a \in \partial\mathcal{C}$ such that $a \in \partial(B_1)$ and $a \notin \partial B_i \ \forall\ i \neq 1$. Here $\partial\mathcal{C}$ denotes the boundary of $\mathcal{C}$. Then there is a path $\gamma : [0,1] \to \partial\mathcal{C}$ such that $\gamma(0) = a = \gamma(1)$.

Proof. We will algorithmically construct the path γ.

Now, since $a \in \partial B_1$ and $a \notin \partial B_i \ \forall\ i \neq 1$, by property of $\mathbb{R}^2$ we can say that there exist two ways from a along the boundary of $\mathcal{C}$. Let us name them positive direction (keeping $\mathcal{C}$ to the right) and negative direction (keeping $\mathcal{C}$ to the left). Let $a = a_0$.

Let us start moving along $\partial\mathcal{C}$ in the positive direction until we meet a point $a_1 \in \partial B_{i_0}$, $B_{i_0} \neq B_1$. From a_1 continue along $\partial B_{i_0} \cap \partial\mathcal{C}$ in the positive direction until we meet a point $a_2 \in \partial B_{i_1}$, $B_{i_0} \neq B_{i_1}$ and proceed in the same fashion. Proceeding in this manner, we get points say $a = a_0, a_1, a_2, ..., a_k, ...$ such that for all $i \neq 0$ a_i is an intersection point of boundary of two balls and that between a_i and a_{i+1} there is no point common to two balls. We denote the arc from a_i to a_{i+1} by $[a_i, a_{i+1}]$. Hence, for all i, there is a path γ_i:[0,1]$\to$ $[a_i, a_{i+1}]$ with $\gamma_i(0) = a_i$ and $\gamma_i(1) = a_{i+1}$. Observe that if the segment $[a_i, a_{i+1}]$ and $[a_j, a_{j+1}]$ have more than two points in common then the segments are equal.

Now, since we are moving along the boundary of finitely many balls we must repeat an arc sometimes during this process. Let k be the least number such that the arc $[a_k, a_{k+1}]$ is either equal to $[a_0, a_1]$, in which case $a_k = a_0$, or contains $[a_0, a_1]$, in which case $a_{k+1} = a_0$. The required path γ is then obtained by joining γ_i successively. Hence our lemma is proved. □

We now proceed to prove the no-retraction theorem for the 2-dimensional case. Let $\mathbb{D}^2$ and S^1 denote, respectively, the closed unit disc and the unit circle.

Theorem 1 *There does not exist a continuous function*

$f : \mathbb{D}^2 \to S^1$ such that f restricted to S^1 is identity.

Proof. We prove the result contradiction. Let there be a retraction map f from $\mathbb{D}^2$ to S^1. Let a, b belong to S^1. Let $A = f^{-1}(a)$ and $B = f^{-1}(b)$.

Since $\{a\}, \{b\}$ are closed in $\mathbb{D}^2$ and since f is continuous, A and B are closed in $\mathbb{R}^2$. Since $\mathbb{R}^2$ is a regular space, around every point of $x \in A$ we can draw a ball $B_{\delta_x}(x)$ whose closure does not intersect B. Also if $x \neq a$ we choose our ball such that $B_{\delta_x}(x)$ does not intersect S^1. Now since A is compact we can fill the set A using finite number of such balls say $B_1, B_2, ..., B_n$.

Note that by our choice the $B_{\delta_a}(a)$ is one of the $B_1, B_2, ...,$ B_n. Let $\partial B_{\delta_a}(a)$ intersect S^1 in a_0, a_1. Let $\mathcal{B} = \bigcup_{i=1}^{n} B_i$.

We now give a path from a_0 to a_1 using $\partial\mathcal{B} \cap \mathbb{D}^2$.

By Lemma 1 starting from a_0 there exists a path along the boundary of $\mathcal{B}$ so that we come back to a_0. Now, there are only two directions on $\partial\mathcal{B}$ to proceed from. One of them is inside $\mathbb{D}^2$ and the other is outside. So the path we constructed had to cut S^1 at some other point other than a_0 which is a_1. So a_0 and a_1 are connected by a path lying inside $\mathbb{D}^2$. Let this path be $\mathcal{P}$. Now, $S^1\backslash\{a, b\} \cup \mathcal{P}$ is connected and the image of this under f is $S^1\backslash\{a, b\}$ because the path avoided A and B.

This is a contradiction as the domain is connected but the image is not. So our assumption of f was wrong. Hence we have proved our theorem. □

We now prove the No-retraction theorem for the three dimension case. For that we begin by stating Lemma 2 below which is a straight forward extension of Lemma 1; in particular, the proof of Lemma 1 is easily adopted to the situation.

Lemma 2 *Let S be a 2-dimensional topological manifold in $\mathbb{R}^3$. Let $B_1, B_2, ..., B_n$ be distinct open balls in $\mathbb{R}^3$*

such that $B_i \cap \mathcal{S}$ is homeomorphic to $\mathbb{D}^2$. Let $U_i = B_i \cap \mathcal{S}$ and $\mathcal{C} = \bigcup_{i=1}^{n} B_i$.

If $a \in \partial\mathcal{C}$ such that $a \in \partial U_1$ and $a \notin \partial U_i \ \forall\ i \neq 1$ then there is a path $\gamma : [0,1] \to \partial\mathcal{C}$ such that $\gamma(0) = a = \gamma(1)$.

Remark 1 *Let $\mathcal{S}$ be a 2-dimensional manifold with boundary in $\mathbb{R}^3$ such that $\partial\mathcal{S}$ is homeomorphic to S^1. Let a, b be two points on the $\partial\mathcal{S}$. Then $\partial\mathcal{S}\backslash\{a,b\}$ consists of two arcs A and B. Let $B_1, B_2, ..., B_n$ be distinct open balls in $\mathbb{R}^3$ such that if $U_i = (B_i \cap \mathcal{S})$, $U_i \cap A = \phi \ \forall\ 1 \leq i \leq n$. Let $\mathcal{C} = \bigcup_{i=1}^{n} U_i$. Then there exists a path γ from a to b lying in $\mathcal{S}$ and avoiding $\mathcal{C}$.*

The reason is as follows:

If $(\mathcal{C} \cap \partial\mathcal{S}) = \phi$ then one of the arcs A or B will give the required path else Lemma 2 and the process described in Theorem 1 can be used to construct the required path. □

Remark 2 *Let $B_1, B_2, ..., B_n$ be 3-dimensional open balls Then $\mathbb{D}^3\backslash(\bigcup_{i=1}^{n} B_i)$ is a 3-dimensional manifold with boun dary. Indeed its boundary is a 2-dimensional topological manifold.*

Now, we have our theorem for the 3-dimensional case.

Theorem 2 *There does not exist a continuous function $f : \mathbb{D}^3 \to S^2$ such that f restricted to S^2 is identity.*

Proof. Let us proceed by contradiction. That is there exists a retraction map, f, from $\mathbb{D}^3$ to S^2. Consider $S^2 = \{(x,y,z) |\ x^2 + y^2 + z^2 = 1\}$. Fix an $\epsilon \in (0,1)$. Let $\hat{A} = \{(x,y,z) \in S^2 |\ z \leq -\epsilon\}$ and $\hat{B} = \{(x,y,z) \in S^2 |\ \epsilon \leq z\}$. Let $A = f^{-1}(\hat{A})$ and $B = f^{-1}(\hat{B})$.

Since $\hat{A}$ and $\hat{B}$ are closed in S^2 and since f is a continuous, A and B are also closed in $\mathbb{R}^3$. As in Theorem 1 since A is compact there exist finitely many number of open balls $B_{z1}, B_{z2}, ..., B_{zn}$ such that $\overline{B_{zi}} \cap B = \phi$ and $\overline{B_{zi}} \cap \{(x,y,z) \in S^2 | \ z \geq 0\} = \phi$.

By Remark 2, $\mathbb{D}^3 \setminus \bigcup_{i=1}^{n} B_{zi}$ is a manifold with boundary and $\mathcal{B}$ denote the boundary. Now $\{(x,y,z) \in S^2 | \ z > 0\} \subset \mathcal{B}$. So, let $\mathcal{B}_z = \mathcal{B} \setminus \{(x,y,z) \in S^2 | \ z > 0\}$. Then $\mathcal{B}_z$ is a 2-dimensional manifold with boundary $\{(x,y,z) \in S^2 | \ z = 0\}$. Also observe that $\mathcal{B}_z \subset f^{-1}\{(x,y,z) \in S^2 | \ |z| < \epsilon\}$.

Similarly, construct a 2-dimensional manifold $\mathcal{B}_x$ as above replacing the role of the coordinate z by x. $\mathcal{B}_x$ has boundary $\{(x,y,z) \in S^2 | \ x = 0\}$.

Claim 1 *There exists a connected component* $\mathcal{P} \subset \mathcal{B}_x \cap \mathcal{B}_z$ *which intersects both* $\{(x,y,z) \in S^2 | x = 0, y < 0\}$ *and* $\{(x,y,z) \in S^2 | x = 0, y > 0\}$.

Proof of Claim 1. If not so, let $\mathcal{X}$ be the union of all those components of $(\mathcal{B}_x \cap \mathcal{B}_z)$ which intersects only $\{(x,y,z) \in S^2 | x = 0, y < 0\}$. $\mathcal{Y} = (\mathcal{B}_x \cap \mathcal{B}_z) \setminus \mathcal{X}$.

By same argument as in Theorem 1 and since $\mathcal{X}$ is compact we can cover $\mathcal{X}$ by finitely many open balls $C_1, C_2, ..., C_s$ such that the requirements of Lemma 2 are satisfied and $\overline{C_i} \cap \mathcal{Y} = \phi$ and (0,0,1),(0,0,-1) $\notin \overline{C_i}$.

Now, by Remark 1, (0,0,1) and (0,0,-1) are connected by a path $\gamma : [0,1] \to \mathcal{B}_x$ avoiding $\mathcal{X}$ and $\mathcal{Y}$ with $\gamma(0) = (0,0,1)$ and $\gamma(1) = (0,0,-1)$. Hence γ does not intersect the boundary of $\mathcal{B}_z$. But this is not possible as (0,0,1) and (0,0,-1) are in two sides of $\mathcal{B}_z$. That is (0,0,-1) $\in \bigcup_{i=1}^{n} B_{zi}$ but (0,0,1) is not and so any path joining (0,0,1) and (0,0,-1) intersects the boundary which is $\mathcal{B}_z$. This is a contradiction. This proves the claim.

By Claim 1, $\mathcal{P} \subset (\mathcal{B}_x \cap \mathcal{B}_z) \subset f^{-1}\{(x,y,z) \in S^2 | \ |x| < \epsilon, |z| < \epsilon\}$ is a connected component intersecting both $\{(x,y,z) \in S^2 | \ |x| < \epsilon, \ |z| < \epsilon, y < 0\}$ and $\{(x,y,z) \in S^2 | \ |x| < \epsilon, \ |z| < \epsilon, y > 0\}$.

So $\mathcal{P} \cup \{(x,y,z) \in S^2 |\ |x| < \epsilon, |z| < \epsilon\}$ is connected but the image under f is $\{(x,y,z) \in S^2 |\ |x| < \epsilon, |z| < \epsilon\}$ (because $\mathcal{P} \subset f^{-1}\{(x,y,z) \in S^2 |\ |x| < \epsilon,\ |z| < \epsilon\}$) which is disconnected. So this is a contradiction and hence our assumption was wrong. This proves our theorem. □

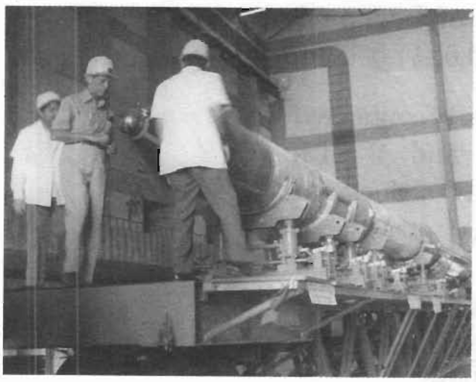
Conclusion

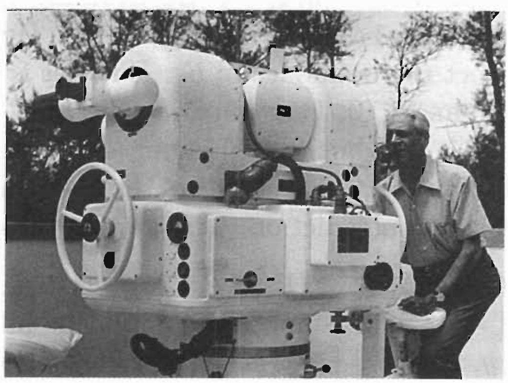
Using the above proof we can prove the Brouwer fixed point theorem and essential mapping theorem for the 2 and 3 dimension cases using only point set topology. Actually one may try to use the ideas to prove the generalised version of no-retraction theorem using elementary topology.

Suggested Reading

[1] Yakar Kannai, An Elementary proof of the no-retraction theorem published in *American Mathematical Monthly,* Vol.88, No.4, 1981.

[2] Robert F Brown, Elementary consequences of the noncontractibility of the circle published in *American Mathematical Monthly,* Vol.81, No.4, 247-252, 1974.





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