Think It Over

This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

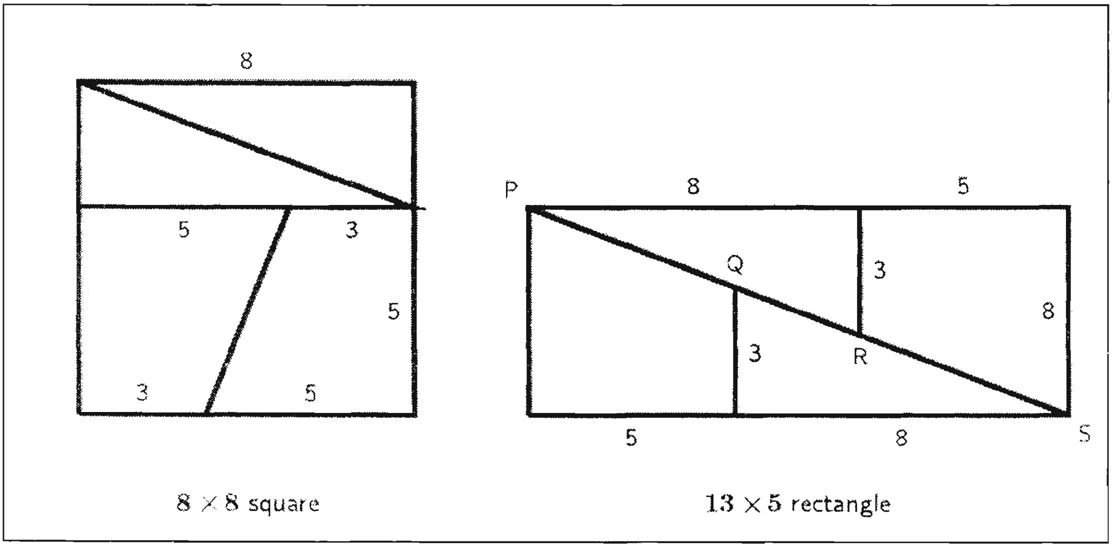
Soubhik Chakraborty
Department of Statistics, Marwari College
Bhagalpur University,
Bhagalpur 812 007, India.

On Rectangling a Square

Is it possible to cut a square into a finite number of pieces and then rejoin the pieces to form a rectangle of greater area?

This looks impossible, yet the following scheme seems to do it. We take a 8×8 square, dissect it as shown in *Figure* 1, then rejoin the pieces to get a 5×13 rectangle. It would appear that an extra unit of area has been created out of nothing. How is this possible? The

Figure 1.



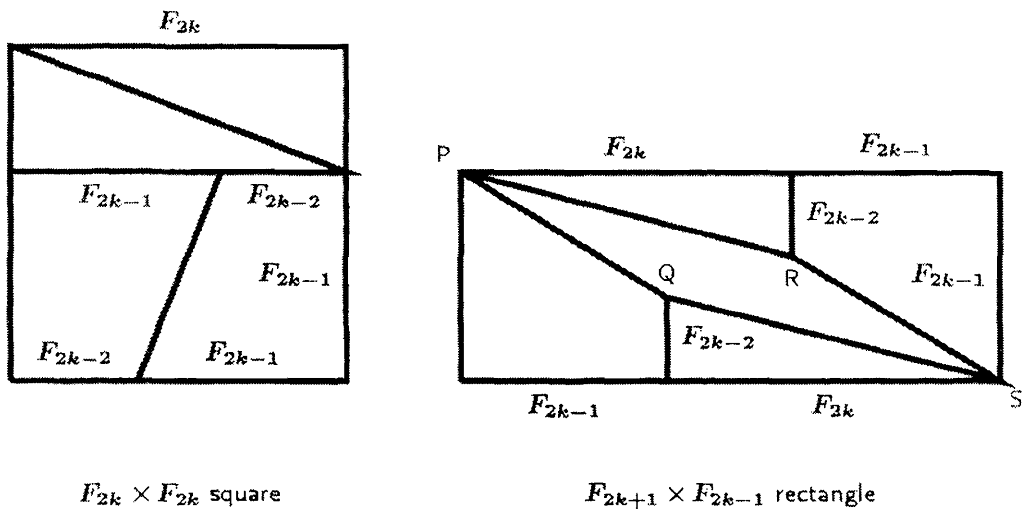


Figure 2.

explanation is that the points P, Q, R, S in the rectangle do not actually lie in a straight line but only appear to do so; in fact, they are the vertices of a parallelogram whose area is unity. This is the source of the 'extra unit.'

To the question, "For which integer sided squares is such a construction possible?" we have a rather fascinating answer: we can accomplish it with any square whose side is the Fibonacci number F_{2k} (for $k > 2$). The corresponding rectangle has dimensions $F_{2k-1} \times F_{2k+1}$, and the source of the 'extra unit' lies in the identity $F_{2k}^2 - F_{2k-1}F_{2k+1} = -1$. The dissection itself is shown in *Figure* 2. (The width of the inner parallelogram is shown highly exaggerated for convenience.)

The area of the parallelogram is unity, so the gaps between its opposite sides are

$$\frac{1}{\sqrt{F_{2k}^2 + F_{2k-2}^2}} \quad \text{and} \quad \frac{1}{\sqrt{F_{2k-1}^2 + F_{2k-3}^2}}.$$

For large k these are very small quantities, and this accounts for the optical effect.

Historical Note

Fibonacci, a contraction of Filius Bonacci ("son of Bonacci"), was the pen name of Leonardo of Pisa, generally regarded as one of the greatest mathematician of his time. It was he who taught the Hindu-Arabic numeration system to Western Europe, through his work *Liber Abaci* which was written in 1202 (the word 'abaci' implies computation). It is ironic that despite his many achievements, Fibonacci is remembered today mainly because of his association with a sequence that he idly introduced in the *Liber Abaci*: "A man put a pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair begets a new pair which from their second month onwards become productive?" A brief reflection will reveal that the solution results in the 'Fibonacci numbers' 1, 1, 2, 3, 5, 8, 13, 21, , in which each number after the second one is the sum of the preceding two numbers.

It turns out that the Fibonacci sequence is richly endowed with properties of various kinds. Some of these have been documented in earlier articles in *Resonance*.

Science is facts; just as houses are made of stone, so is science made of facts; but a pile of stones is not a house, and a collection of facts is not necessarily science.

Jules Henri Poincaré
(1854-1912)
French Mathematician