In this section of Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.*

What is Fishy about the Fish's Eye?

S Chatterjee
Indian Institute of Astrophysics
Bangalore 560 034, India.

Introduction

Maxwell's fish's eye problem is important because it addresses some important issues on the focusing properties optical instruments. The common instruments like lenses, prisms, etc. are the ones where the refractive index changes abruptly at an interface between two media, e.g. air and glass. Maxwell addressed the issue whether there can be focusing effects in media where the refractive index varies continuously, such media being called stratified media.

The relation with the fish's eye appears as follows. Unlike the human eye, which generally receives light traveling from air (except in the case of underwater swimming) the fish receives rays from water (i.e. a medium of fairly high refractive index, ≈ 1.33) which must be focused on a tiny area, like the fish's retina. Maxwell asked the question: if the fish's eye be made of a medium, whose refractive index falls radially from the retinal centre, can it help in focusing on the fish's retina? Maxwell was prompted to tackle the problem since the crystallinity of the fish's retina was known. As it turned out, the prescription that Maxwell followed for focusing gave a refractive index profile, which enabled perfect focusing. Note, in the case of prism and lenses the rays of light get bent only at the interface of separation between media but for a continuously stratified medium the direction and curvature of the rays must change continuously in space. Rays being curves in space along which the radiant energy flows, the question that is to be asked is, what is the equation for such a curve in terms of the refractive index profile? Maxwell published this paper as a short communication in the *Cambridge and Dublin Mathematical Journal*, in February 1854.

Ray Optics and Mathematical Preliminaries

For the fish's eye problem Maxwell established his results by simple geometrical constructions. He mainly used results concerning properties of circles, since he considered the rays in the fish's eye to be circles, a happy conjecture, which gave exact results. Here we follow an approach (method followed in Born and Wolf) that is different from Maxwell's original one but the mathematics that follows must be intelligible to any student of BSc. The equations that we introduce will also help students to tackle problems of more generalized types. We begin by defining the tangent **s** and curvature **K** vectors for a curve in space. Let us consider any point $P(x,y,z)$ on a curve curve OPP′P″ and define the parameter '*l*' as the length of the curve as measured from a fixed point O on it. The location of the point P may be defined by the vector **r**, as shown in *Figure* 1a. We thus have

$$\Delta l = \sqrt{[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]}$$

and we define

$\mathbf{s} = \mathrm{d}\mathbf{r}/\mathrm{d}l$ and $\mathbf{K} = \mathrm{d}\mathbf{s}/\mathrm{d}l = \kappa\,|\mathbf{K}| = \kappa/R, \kappa$ being the unit vector along **K** and R being called the radius of curvature of the curve.

It is seen that

$$s^2 = \mathbf{s}.\mathbf{s} = \underset{\Delta l \to 0}{\mathrm{Lt}}\,[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2]\,/(\Delta l)^2 = (\Delta l)^2\,/(\Delta l)^2 = 1,$$

by using Pythagoras' theorem, which also implies that $\mathrm{d}s^2/\mathrm{d}l = 0 = \mathbf{s}.\mathrm{d}\mathbf{s}/\mathrm{d}l = \mathbf{s}.\mathbf{K}$, i.e. **s** and κ are perpendicular to each other at

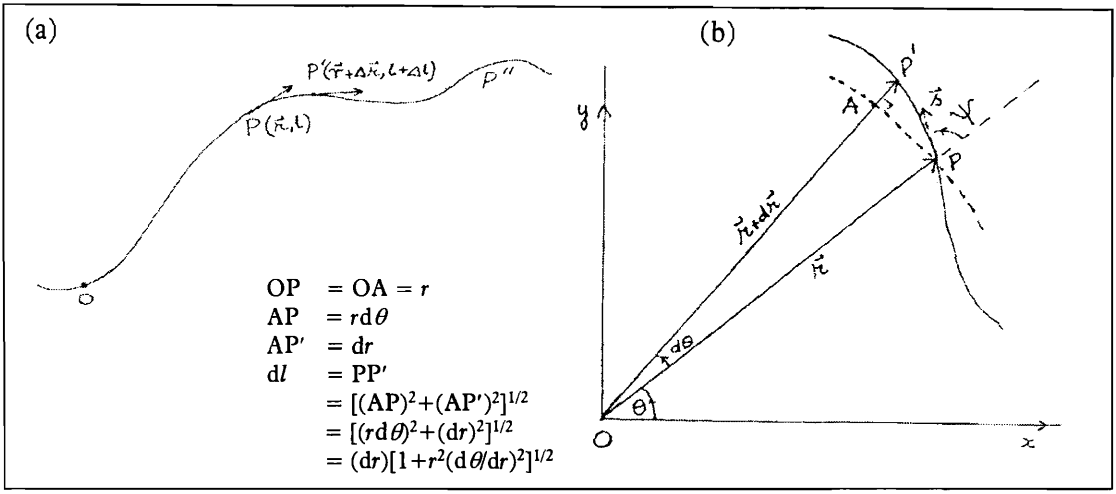

Figure 1. Concerning the calculus of curves.

every point on the curve. Further defining a unit vector $\mathbf{t} = \mathbf{s} \times \kappa$, it is possible to form a triad, consisting of vector **t**, **s**, κ at every point $P(x, y, z)$ of the curve. There are some important relations between the above vectors, which are not of concern to us here. It is also clear that by following **s** we actually follow the curve while the changes in its direction follow the curvature of the curve.

The question that we ask is: if there is an inhomogeneous medium, i.e. in which the refractive index $n(\mathbf{r})$ depends on the position **r** in space, then what is the equation to the ray, along which the radiant energy of light flows? The answer is given by (1), given below, that follows from Fermat's principle, (we take this equation for granted as we need the calculus of variations to derive this from the Fermat principle)

$$\mathrm{d}(n(\mathbf{r})\mathbf{s})/\mathrm{d}l = \mathbf{grad}\; n(\mathbf{r}) \qquad (1)$$

The lhs (left hand side) of (1) can be expanded as,

$$\begin{aligned}\mathrm{d}(n(\mathbf{r})\mathbf{s})/\mathrm{d}l &= n(\mathbf{r})[\mathrm{d}\mathbf{s}/\mathrm{d}l] + \mathbf{s}[\mathrm{d}n(\mathbf{r})/\mathrm{d}l]\\ &= n(\mathbf{r})[\mathrm{d}\mathbf{s}/\mathrm{d}l] + \mathbf{s}[(\partial n/\partial x)(\mathrm{d}x/\mathrm{d}l) + (\partial n/\partial y)(\mathrm{d}y/\mathrm{d}l)\\ &\qquad + (\partial n/\partial z)(\mathrm{d}z/\mathrm{d}l)]\\ &= n(\mathbf{r})[\mathrm{d}\mathbf{s}/\mathrm{d}l] + \mathbf{s}\,[n_x s_x + n_y s_y + n_z s_z\,]\\ &= n(\mathbf{r})[\mathrm{d}\mathbf{s}/\mathrm{d}l] + \mathbf{s}\,[\mathbf{s}.\mathbf{grad}\; n(\mathbf{r})] \qquad (2)\end{aligned}$$

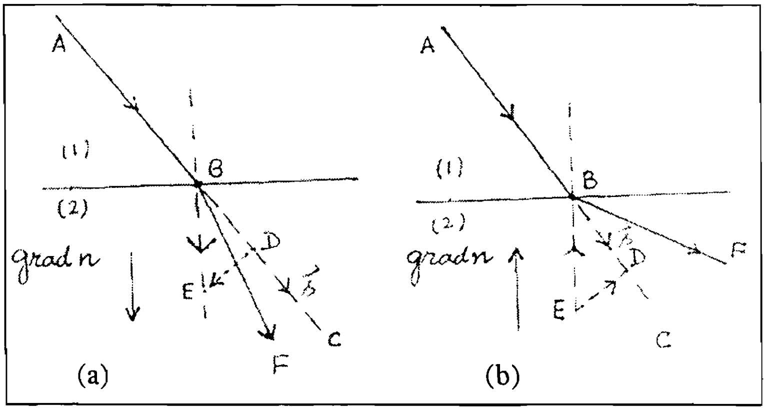

Figure 2. On the geometry of rays in a stratified medium.

so that combining (1) and (2) we get the equation for the curvature for the ray in an inhomogeneous medium to be,

$$\mathrm{d}\mathbf{s}/\mathrm{d}l = [\mathbf{grad}\, n(\mathbf{r}) - \mathbf{s}\ (\mathbf{s}.\mathbf{grad}\, n(\mathbf{r}))]/n(\mathbf{r}) \qquad (3)$$

Equation (3) can also be written as

$$\mathbf{s}(l + \mathrm{d}l) = \mathbf{s}(l) + \mathrm{d}l[\ \mathbf{grad}\, n(\mathbf{r}) - \mathbf{s}\ (\mathbf{s}.\mathbf{grad}\, n(\mathbf{r}))]/n(\mathbf{r}), \qquad (4)$$

where the last term in (4) shows the direction in which the ray swerves in traveling a distance dl. Since $\mathbf{s}(\mathbf{s}.\mathbf{grad}\ n(\mathbf{r}))$ is the component of grad $n(\mathbf{r})$ along **s**, the term $[\mathbf{grad}\ n(\mathbf{r}) - \mathbf{s}(\mathbf{s}.\mathbf{grad}\ n(\mathbf{r}))]/n(\mathbf{r})$ is a vector perpendicular to **s**, which is not surprising since $\mathbf{s}.\mathrm{d}\mathbf{s}/\mathrm{d}l = 0$, i.e. a ray always swerves in a direction perpendicular to itself. This can be understood from the *Figures* 2a and 2b.

In both the *Figures* 2a and b, the ray **AB** goes from medium 1 to 2 where the path length ABF is considered to be an infinitesimal dl. In *Figure* 2a, the ray comes from a rarer to a denser one while the reverse is the case in *Figure* 2b. Thus in *Figure* 2a the vector $\mathbf{grad}(n(\mathbf{r}))$ acts along **BE**, while it acts along **EB** in *Figure* 2b. The initial ray direction acts along **BC** in both the cases so that $(\mathbf{s}.\mathbf{grad}\ n(\mathbf{r}))$ acts along **BD** in *Figure* 2a and along **DB** in *Figure* 2b. Thus from the triangle of vectors it is seen that $[\mathbf{grad}\ n(\mathbf{r}) - \mathbf{s}(\mathbf{s}.\mathbf{grad}\ n(\mathbf{r}))]/n(\mathbf{r})$ acts along **DE** in *Figure* 2a and along **ED** in *Figure* 2b, both being in the plane of the paper. Thus in *Figure* 2a the rays turn in the direction **DE**, i.e. towards the normal to the surface of separation, while in 2b it turns away from the surface to the surface of separation. These qualitative ideas, which are taught in school level course in the chapter on light are all contained in (2) and (4), from which it also follows that,

$$\begin{aligned} \mathrm{d}(\mathbf{r} \times n\mathbf{s})/\mathrm{d}l &= (\mathrm{d}\mathbf{r}/\mathrm{d}l)(n\mathbf{s}) + \mathbf{r} \times \mathrm{d}(n\mathbf{s})/\mathrm{d}l \\ &= n\mathbf{s} \times \mathbf{s} + \mathbf{r} \times \mathbf{grad}(n(\mathbf{r})) \\ &= \mathbf{r} \times \mathbf{grad}(n(\mathbf{r})) \end{aligned} \qquad (5)$$

We now apply (2-5) in a special situation, where the refractive index $n(\mathbf{r}) = n(r)$, i.e. has only radial dependence, being independent of the angles θ, φ. We then find $\mathbf{grad}(n(\mathbf{r}))=(\mathbf{r}/|\mathrm{r}|)\, \partial n/\partial \mathrm{r}$, which being directed along $\mathbf{r}$ makes the last term of (5) also as zero. In such a case $\mathrm{d}(\mathbf{r} \times n\mathbf{s})/\mathrm{d}l = 0$, on being integrated gives,

$$\mathbf{r} \times n\mathbf{s} = \mathbf{C} \qquad (6)$$

along a ray, where $\mathbf{C}$ is a constant vector.

We note that (6) being a vector equation, the direction of $\mathbf{r} \times n\mathbf{s}$ must not change and be directed along the same direction at every point on the ray, which also means that $\mathbf{r}$ and $\mathbf{s}$ always lie on the same plane. Thus the initial $\mathbf{r}$ and $\mathbf{s}$ determine the common plane in which they always lie. Using this plane as our reference plane, we can describe the ray by only r and θ, the angle φ being now unimportant.

Further, the magnitude $|\mathbf{C}| = |n\mathbf{r} \times \mathbf{s}| = C$ must also not change along the ray. From *Figure* 1b and from the definition of the cross product we find

$$nr \sin\psi = C, \qquad (7)$$

where ψ is the angle between $\mathbf{r}$ and $\mathbf{s}$. From *Figure* 1b we also find that the length $\mathrm{d}l$ of the curve follows

$$\mathrm{d}l = \sqrt{[(\mathrm{d}r)^2 + r^2 (\mathrm{d}\theta)^2]} \qquad (8)$$

so that we find

$$\mathbf{s} = \mathrm{d}\mathbf{r}(l)/\mathrm{d}l = \mathrm{d}\mathbf{r}(l)/[(\mathrm{d}r)\,[1 + r^2 (\mathrm{d}\theta/\mathrm{d}r)^2]^{1/2}]. \qquad (9)$$

The *Figure* 1b further shows, that **s** being a unit vector along the tangent at P, its component along **r** (i.e. OP) is simply cos ψ, i.e.

$$s_r = dr/[dr[1 + r^2 (d\theta/dr)^2]] = \cos\psi \tag{10}$$

or $$\sin^2 \psi = 1 - \cos^2\psi = r^2 /[r^2 + (dr/d\theta)^2] \tag{11}$$

so that (7) on using (11) reads

$$nr^2 /[1 + r^2 (d\theta/dr)^2] = C \tag{12}$$

or $$dr/d\theta = (r/C)[n^2r^2 - C^2]^{1/2} \tag{13}$$

at any point P(r, θ) on the ray.

On integrating along the ray we find

$$\theta(r) = C \int dr/[r[n^2 r^2 - C^2]^{1/2}] , \tag{14}$$

the lower limit of integration being any $r = r_0$ while the upper limit is $r = r$.

Optics of the Fish Eye

It is here that we can introduce Maxwell's prescription of the fish's eye optics. He conjectured that the rays in the fish's eye trace out circles and that all the rays emanating from any point P_0 must cross the same given point P. This is the condition for perfect focusing, i.e. all rays from P_0, irrespective of their initial direction of propagation converge to the same point P. Maxwell came to the conclusion that for this to happen the refractive index profile must follow (15) given below (see problem 1). We follow here a converse of what Maxwell did, i.e. by assuming the profile to be as given in (15), we show that we obtain perfect focusing, the rays being circles.

If we consider the refractive index profile of the medium to be

$$n(r) = n_0/[1 + r^2/a^2] \tag{15}$$

we get on defining the dimensionless quantities $Q = C/(an_0)$ and $\rho = r/a$, with some simple algebra,

$$\theta(r) = \int Q(1 + \rho^2)d\rho / [\rho[\rho^2 - Q^2(1 + \rho^2)^2]^{1/2}]. \quad (16)$$

Using the identity,

$$[d\{\sin^{-1}(Q(\rho^2 - 1)/(\rho[1 - 4Q^2]^{1/2}))\}/d\rho] =$$
$$[Q(1 + \rho^2)]/[\rho[\rho^2 - Q^2(1 + \rho^2)^2]^{1/2}] \quad (17)$$

we get on integrating of (16)

$$\theta(\rho) = \sin^{-1}(Q(\rho^2 - 1)/(\rho[1 - 4Q^2]^{1/2})) + \alpha, \quad (18)$$

where α is a constant of integration. Equation (18) thus leads us to

$$\sin(\theta - \alpha)[\rho/[\rho^2 - 1]] = Q/[1-4Q^2] = C/[a^2 n_0^2 - 4C^2]^{1/2}. \quad (19)$$

Suppose the ray passes through any point $P_0(r_0,\theta_0)$, then we have from (18)

$$\alpha = \theta_0 - \sin^{-1}[(C/(ar_0))(r_0^2 - 1)/[a^2 n_0^2 - 4C^2]^{1/2}], \quad (20)$$

so that for different rays passing through the same $P_0(r_0, \theta_0)$ the values of C and hence α are different, or in other words, C and α distinguish the different rays. From (19) it is easy to see that

$$\frac{r^2 - a^2}{ar\sin(\theta - \alpha)} = \frac{r_0^2 - a^2}{ar_0\sin(\theta_0 - \alpha)} = \frac{\left[a^2 - n_0^2 - 4C^2\right]^{1/2}}{C} \quad (21)$$

giving the equation for any ray passing through $P_0(r_0, \theta_0)$ for any given α.

It is easy to verify that (21) is also satisfied by $P(r_1 = a^2/r_0, \theta_1 = \pi + \theta_0)$ irrespective of C or α. This implies that any ray passing through $P_0(r_0,\theta_0)$ passes through $P(r_1,\theta_1)$ irrespective of their initial direction, i.e. P is the focal point for all rays leaving P_0 or vice versa. Note that r_1 *depends only on 'a' and 'r_0' but not on* n_0. This point of focus being unique for all rays, the fish's eye is thus seen to exhibit perfect focusing.

Let us now examine the trajectory of the rays in a fish's eye.

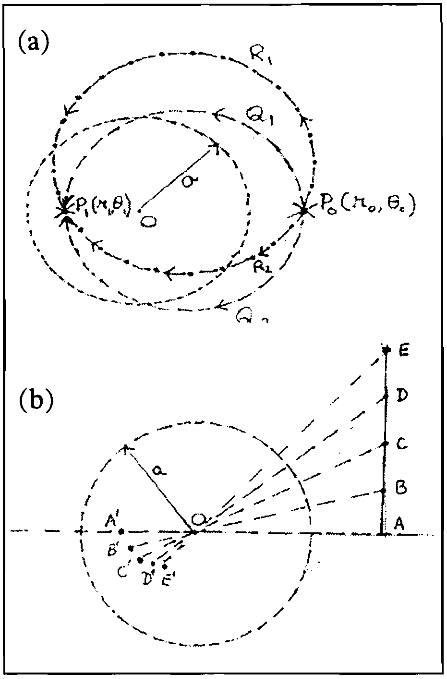

Figure 3. Ray tracings in a fish's eye.

Expanding $\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$ and noting that in terms of (r,θ) the Cartesian coordinates are given by $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = r^2$, we have

$$(x + b\sin\alpha)^2 + (y - b\cos\alpha)^2 = a^2 + b^2, \qquad (22)$$

where

$$b^2 = a^2[a^2n_0^2 - 4C^2]/(4C^2). \qquad (23)$$

Equation (22) shows that all the rays from P_0 (r_0, q_0) trace circles with centres at $(-b\sin\alpha, + b\cos\alpha)$ and radii $(a^2 + b^2)^{1/2}$ This is schematically shown in *Figure* 3a.

It is seen that in the fish's eye, focusing is perfect but is the image defect free? It is indeed not so as is seen from *Figure* 3b. Let us consider a straight object ABCDE, perpendicular to OA, O being the centre of the fish's eye Then A is focused at A′, where OA′= a^2/OA, while B is focused at B' where OB′ = a^2/OB. Similarly, C, D, E are focused at C′, D′, E′. Thus the image A′B′C′D′E′ is inverted, but the image is not parallel to the object ABCDE. The magnification at every point is given by $m = -r_1/r_0$ which is not uniform throughout the image. The image is thus not defect free, details of which are left to an interested reader to analyze.

To conclude, let us see where does the fish's eye focus a distant object. Making $r_0 >> a$ (we use this as a criterion for a distant object) we find that the focusing takes place at $r_1 = a^2/r_0 << a$ i.e. within the fish's retina, $r < a$. This was the problem that we had set out to investigate.

Later Developments

Though the above model is a simplified one, Maxwell's fish's eye problem shows how a continuous refractive index profile helps to produce a focused image. These ideas received considerable thrust since the mid 1950's when technological advance allowed the manufacture of dielectric media with continuously varying refractive index. Modern fibre optics use such media where the refractive index has radial gradation for a fibre of cylindrical symmetry. The work initiated by Maxwell was enormously enriched by Luneberg, ninety years after Maxwell published the fish's eye problem. Luneberg found that in a system with $n(r) = [2-(r/a)^2]^{1/2}$, where $r < a$, every pencil is brought to a sharp focus. This Luneberg lens is an important device in many systems, including microwave antennas, where focusing has to be achieved over wide angles. Luneberg's work, first presented as lectures in the Brown University in 1944 remained in obscurity for about a decade. His lecture notes were first published in 1964, i.e. sixteen years after the author's death. Indeed the optics of light was gaining rapid advance at that time but the optics of electrons, e.g. in electron microscopy too was becoming important to the scientist and technologist alike. Within certain limits the ray constructions applied to both. Hence by the mid 1950s Luneberg's work gained recognition. The Maxwell fish's eye thus got the most appropriate centenary commemoration.

Problem 1. Considering (after Maxwell) that the rays in the fish's eye trace out circles, we consider the ray to be described by $r^2 - 2par\sin(\theta-\alpha) - a^2 = 0$. Show that this curve passes through the points P_0 and P, described in the text for all values of the parameters p and α. Using (12) show that the refractive index profile must then be as in (15), so that p is related to C as, $n_0^2 = 4C^2(1+p^2)$.

Problem 2. Using a profile $n(z) = n_0(1 + \alpha z)$, use (1) and (2) to obtain the equation of a ray of light shot at an angle α to the horizontal (x-axis) from a point (x_0, z_0). Does this help to understand the optics of mirages?