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Sum of Powers of Natural Numbers using Integration

Introduction

We know that the sum of the first n natural numbers is $n(n+1)/2$. This can be proved easily in several ways. For example, if S denotes the sum of first n natural numbers, then $S = \sum_{i=1}^{n} i = \sum_{i=1}^{n}(n+1-i)$ and therefore $2S = \sum_{i=1}^{n}(i+n+1-i) = n(n+1)$ which gives $S = n(n+1)/2$. One can also use the principle of mathematical induction to prove the result. Here we present a proof of the result using integration. Let $S_k(n) = \sum_{i=1}^{n} i^k$ be the sum of kth powers of first n natural numbers. Let

$$\begin{aligned} p_1(x) &= x + \frac{1}{2} \\ p_2(x) &= x^2 + x + \frac{1}{6} \\ p_3(x) &= \frac{4x^3 + 6x^2 + 2x}{4}. \end{aligned}$$

Then

$$\int_{k-1}^{k} p_1(x)dx = k \quad \text{and} \quad \int_{0}^{n} p_1(x)dx = \frac{n(n+1)}{2},$$

and

$$\begin{aligned} S_1(n) &= 1 + 2 + \quad + n \\ &= \int_0^1 p_1(x)dx + \int_1^2 p_1(x)dx + \quad + \\ &\qquad \int_{n-1}^{n} p_1(x)dx \\ &= \int_0^n p_1(x)dx = \frac{n(n+1)}{2}. \end{aligned}$$

Similarly we have

$$\int_{k-1}^{k} p_2(x)dx = k^2 \quad \text{and}$$

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$$\int_0^n p_2(x)dx = \frac{n(n+1)(2n+1)}{6}.$$

Therefore we have

$$\begin{aligned} S_2(n) &= 1^2 + 2^2 + \cdots + n^2 \\ &= \int_0^1 p_2(x)dx + \cdots + \int_{n-1}^n p_2(x)dx \\ &= \int_0^n p_2(x)dx = \frac{n(n+1)(2n+1)}{6}. \end{aligned}$$

Also we have

$$\begin{aligned} S_3(n) &= 1^3 + 2^3 + \cdots + n^3 \\ &= \int_0^1 p_3(x)dx + \cdots + \int_{n-1}^n p_3(x)dx \\ &= \int_0^n p_3(x)dx = \frac{[n(n+1)]^2}{4}. \end{aligned}$$

These examples suggest the problem of finding a function $p_k(x)$ such that

$$i^k = \int_{i-1}^i p_k(x)dx.$$

This function $p_k(x)$ can be used to represent the sum of the kth powers of the first n natural numbers as an integral:

$$1^k + 2^k + \cdots + n^k = \int_0^n p_k(x)dx.$$

In this article, we show that such a polynomial $p_k(x)$ exists and we prove a recurrence relation satisfied by $p_k(x)$. It turns out from this recurrence relation that $p'_k(x) = kp_{k-1}(x)$. Also, we prove that the sum of kth powers of the first n natural numbers is a polynomial of degree $k+1$ without constant term and obtain a nice relation between the sum of powers of natural numbers and the Vandermonde determinant.

The sum of k th powers of the first n natural numbers is a polynomial of degree k+1 without constant term.

Sum of Powers by Integration

Let us start with the assumption that such a polynomial $p_k(x)$ exists for each k and analyse what conditions they must satisfy. As we shall see, these conditions allow us to actually define the polynomials. Since

$$(i+1)^{k+1}-i^{k+1} = \binom{k+1}{1} i^k + \binom{k+1}{2} i^{k-1} + \cdots + \binom{k+1}{k+1}$$

we have

$$(k+1)\int_{n-1}^{n} (x+1)^k dx = \sum_{r=1}^{k+1} \binom{k+1}{r} \int_{n-1}^{n} p_{k+1-r}(x)dx.$$

Therefore, it is clear that we may actually *define* the polynomials $p_k(x)$ recursively by

$$p_k(x) = (x+1)^k - \frac{\binom{k+1}{2}p_{k-1}(x) + \cdots + \binom{k+1}{k+1}p_0(x)}{k+1}.$$

This recurrence relation along with $p_0(x) = 1$ can be used to find $p_k(x)$ for any $k > 0$. For example, we have

$$\begin{aligned}
p_1(x) &= x + \frac{1}{2} \\
p_2(x) &= x^2 + x + \frac{1}{6} \\
p_3(x) &= \frac{4x^3 + 6x^2 + 2x}{4} \\
p_4(x) &= x^4 + 2x^3 + x^2 - \frac{1}{30} \\
p_5(x) &= \frac{6x^5 + 15x^4 + 10x^3 - x}{6} \\
p_6(x) &= x^6 + 3x^5 + \frac{5}{2}x^4 - \frac{x^2}{2} + \frac{1}{42}.
\end{aligned}$$

In fact, it follows that p_k is a polynomial of degree k and has rational coefficients. Notice that $p'_n(x) = np_{n-1}(x)$ for $n = 1, 2, \ldots, 6$. In general we can easily prove that it is true for any integer n by using induction.

The polynomials p_n are closely related to what are known in the literature as Bernoulli polynomials B_n; indeed $p_n(x) = B_n(x+1)$.

Vandermonde Determinant and Sum of Powers

Consider the equations

$$(i+1)^{k+1} - i^{k+1} = \binom{k+1}{1} i^k + \binom{k+1}{2} i^{k-1} + \cdots + \binom{k+1}{k+1}$$

for $i = 1, 2, \ldots, n$. On adding them, we get

$$(n+1)^{k+1} - 1 = \binom{k+1}{1} S_k(n) + \binom{k+1}{2} S_{k-1}(n) + \cdots + \binom{k+1}{k+1} S_0(n).$$

As $S_0(n) = n$, this equality proves easily by induction on k that $S_k(n)$ is a polynomial in n of degree $k+1$ without constant term. Also the coefficient of n^{k+1} in $S_k(n)$ is $\frac{1}{k+1}$. This can be proved from the fact that $p_k(x)$ is a polynomial of degree k with the coefficient of x^k is one by using the integral representation $S_k(n) = \int_0^n p_k(x)dx$.

Since $S_k(n)$ is a polynomial in n of degree $k+1$ without constant term, we can write

$$S_k(n) = a_1 n + a_2 n^2 + a_3 n^3 + \cdots + a_{k+1} n^{k+1}$$

One can find the coefficients a_i by substituting the values $n = 1, \ldots, k+1$ and solving the corresponding matrix equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ 2 & \cdots & 2^{k+1} \\ \vdots & & \vdots \\ (k+1) & \cdots & (k+1)^{k+1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k+1} \end{pmatrix} = \begin{pmatrix} S_k(1) \\ S_k(2) \\ \vdots \\ S_k(k+1) \end{pmatrix}$$

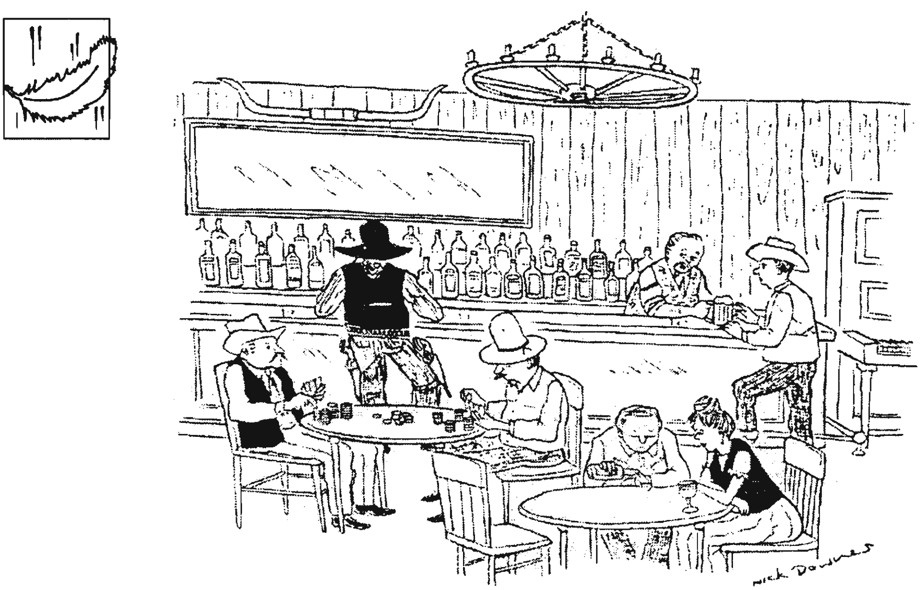
Note that the determinant of the coefficient matrix M is $(k+1)!$ times the determinant of

$$\begin{pmatrix} 1 & 1 & & 1 \\ 1 & 2 & & 2^k \\ \vdots & \vdots & & \vdots \\ 1 & (k+1) & & (k+1)^k \end{pmatrix}$$

which is nonzero; the latter factor is the Vandermonde determinant. As we have

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k+1} \end{pmatrix} = M^{-1}S,$$

this gives an interesting connection between the sum of kth powers of the first n natural numbers and the Vandermonde determinant.



That's Dead-Eye Dan, known far and wide for his fast gun, mean temper and extra y chromosome.

From: Gene Antics