Supersymmetry

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During the last few decades, theoretical physicists have introduced *symmetries* (which may or may not have any geometrical interpretation) with the aim of solving difficult problems. In this article, we shall first present the salient features of one such symmetry called *supersymmetry*. Then, we shall show the power of supersymmetry in tackling quantum mechanical systems described by non-trivial potentials.

Introduction

Symmetries play a significant role in reducing the number of degrees of freedom of any complex system. Hence, some features of a complex system can be determined by exploiting the symmetry properties of the system. In fact, the theory describing unifications of the fundamental particles and the strong, electro-weak forces of nature, called standard model, is based on such symmetry principles [1,2]. Throughout the discovery of all those symmetries, physicists always maintained that *fermions and bosons were two different classes of particles*. Such a distinction between bosons and fermions poses certain problems in theoretical models. For instance, even though the standard model could explain most of the experimental observations about the elementary particles, there are some shortcomings of the model. For example, the Higgs scalar in the standard model is still undiscovered. Further, there are problems with the Higgs scalar, viz., we obtain correction terms to its mass which are infinite (*usually called divergence*). This unphysical divergence can be corrected by fine tuning, but it is an unpleasant feature of the standard model. Therefore, we require a completely non-traditional approach to remove such divergences.

Keywords
Symmetry, elementary particles, quantum mechanics.

This led theoretical physicists to come up with an innovative idea of introducing a novel symmetry called *supersymmetry* which resulted in a finite mass for the Higgs scalar. Unlike traditional symmetries, supersymmetry does not treat bosons and fermions as two different classes of particles. The supersymmetry operation converts bosons into fermions and vice versa. Thus, supersymmetry introduces a fermionic (bosonic) partner for every boson (fermion) differing by half a unit of spin quantum number. The partner is called the superpartner (sparticle) of the original particle. The particle and its superpartner must be identical in all quantum numbers except the spin quantum number.

The theoretical proposition of supersymmetry needs to be experimentally tested, i.e., sparticles of the elementary particles must be observed in high-energy collider experiments. Present day accelerators work at energies much below the energy needed for testing supersymmetry. Even though it is still a debatable question whether nature possesses supersymmetry or not, there are interesting applications of supersymmetry. In fact, we will show that quantum mechanical systems involving nontrivial potentials can be easily solved by exploiting the properties of supersymmetry.

The plan of this article is as follows: First we briefly describe the meaning of symmetry. Then we discuss bosonic and fermionic quantum harmonic oscillators, which are combined with the next section to show the natural emergence of a supersymmetric oscillator. Later we present the formalism of obtaining any supersymmetric system from two partner systems. Using such a supersymmetry formalism, we relate the famous example of a particle in an infinite potential well, which is an exactly solvable system, to a partner system described by a non-trivial potential.

Supersymmetry introduces a fermionic (bosonic) partner for every boson (fermion).

What is a Symmetry?

A *symmetry* of an object (system) is any transformation under which that object (system) remains *invariant*, i.e., object remains the same before and after the transformation. For example, consider a square. It is invariant under reflection about a diagonal. So, reflection about a diagonal is a symmetry of a square. In physical situations, there can be other symmetries which may not have such a geometrical interpretation. Hence, to describe the symmetries (both geometrical as well as non-geometrical) possessed by physical systems, it will be useful to know a quantitative definition.

A formal definition of symmetry is as follows:

[1] An operator which determines energy states of the system.

For a system of particles, a symmetry is any transformation under which the *Hamiltonian*[1] describing the system remains invariant. Such a symmetry transformation is generated by an operator usually referred to as *generator*. A system is said to possess a symmetry if the Hamiltonian *commutes* with the generator of the symmetry transformation. The mathematical meaning of the words 'generator, commutes' will be obvious from the following example.

Consider *translation* as the symmetry transformation. If $\psi(\vec{r})$ is the wave function of the system before translation, and $\psi(\vec{r}+\vec{a})$ is the wave function after translation by a constant vector $\vec{a}$, then

$$\psi(\vec{r}+\vec{a}) = e^{i\vec{a}\cdot\vec{p}/\hbar}\ \psi(\vec{r}),$$

where $\vec{p} = -i\hbar\nabla$ is the momentum operator. The above equation is nothing but a Taylor series expansion of the wave function. The momentum operator $\vec{p}$ is called the *generator* of translations because it generates the wave function at $\vec{r}+\vec{a}$ (coordinate after transformation) from the wave function at $\vec{r}$ (coordinate before transformation). We know that if the potential V is constant in space, then the Hamiltonian $H = -\frac{\hbar^2\nabla^2}{2m} + V$ is invariant under translation. Also, if V is constant, then the commutator bracket $[H,\vec{p}] = H\vec{p} - \vec{p}H$ is zero, i.e., H commutes with $\vec{p}$. We will now see the consequence of the zero commutator bracket on the wave function.

Consider the wave function $\psi(\vec{r})$ satisfying the following eigenvalue equation:

$$H\psi(\vec{r}) = E\psi(\vec{r}),$$

where E is the energy eigenvalue. Then, the zero commutator bracket operating on $\psi(\vec{r})$ gives

$$H\,[\vec{p}\psi(\vec{r})] = \vec{p}H\,\psi(\vec{r}) = E[\vec{p}\psi(\vec{r})]$$

Thus, we see that for a given energy E, there are two eigenfunctions $\psi(\vec{r})$ and $\vec{p}\psi(r)$. These wave functions are usually referred to as *degenerate eigenfunctions.*

Our aim is to find a system which is invariant under the new symmetry transformation, called supersymmetry transformation, and also determine the explicit form for the supersymmetric generator. The first step in this direction will be to study two simple systems – a bosonic and a fermonic harmonic oscillator, which are the building blocks for constructing a supersymmetric oscillator. Then, we can study the properties of supersymmetry which convert bosons into fermions.

Harmonic Oscillator

We shall now study some aspects of bosonic and fermionic oscillators.

The Hamiltonian H for a simple harmonic oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2,$$

where m is the mass and k is the spring constant. We can introduce a dimensionless variable $y = \sqrt{\frac{m\omega}{\hbar}}x$, where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency. Then, H can be factorised as

$$H = \frac{\hbar\omega}{2}\left(aa^\dagger + a^\dagger a\right), \tag{1}$$

where $a^\dagger = -\frac{d}{dy} + y$ and $a = \frac{d}{dy} + y$. The energy eigenstates which are solutions of the Schrödinger equation are the Hermite polynomials

$$\psi_n(y) = \langle y|n\rangle \propto e^{-y^2/2}\; H_n(y),$$

where $|n\rangle$'s are usually referred to as number states, with n denoting the excited level of the harmonic oscillator or the number of particles. The operators a and $a^\dagger$ are called annihilation and creation operators because their action on the number states is as follows:

$$a^\dagger|n\rangle \;=\; \sqrt{n+1}|n+1\rangle, \qquad (2)$$

$$a|n\rangle \;=\; \sqrt{n}|n-1\rangle,\; a|0\rangle \;=\; 0. \qquad (3)$$

In other words, a ($a^\dagger$) lowers (raises) the state $|n\rangle$ to $|n-1\rangle$ ($|n+1\rangle$).

Bosons and fermions are distinguished by Pauli's exclusion principle. That is, no two fermions can exist in the same quantum state, whereas any number of bosons can exist in the same quantum state. This allows the number of particles n in a quantum state $|n\rangle$ to take any value for a bosonic oscillator, whereas for a fermionic oscillator, n must be either zero or one. We shall see in the following subsections how such a distinction is implemented.

Bosonic Harmonic Oscillator

We shall denote the operators and the number states of the bosonic harmonic oscillator with a subscript b to remember their *bosonic* nature. The operators $a_b, a_b^\dagger$ satisfy

$$[a_b, a_b^\dagger] = 1,\; [a_b, a_b] \;=\; 0, [a_b^\dagger, a_b^\dagger] \;=\; 0. \qquad (4)$$

Using the above commutation relation in (1), we can write the Hamiltonian for the bosonic oscillator H_b as

$$H_b = \hbar\omega_b\left(a_b^\dagger a_b + \frac{1}{2}\right), \qquad (5)$$

whose energy eigenvalues E_{n_b} are obtained from

$$H_b|n_b\rangle = E_{n_b}|n_b\rangle, \text{ where } E_{n_b} = (n_b + 1/2)\hbar\omega_b \ , \quad (6)$$

where $n_b = 0, 1, 2, \quad \infty$. It is appropriate to point out that the commutator bracket does not impose any restriction on the number of particles n_b, and hence plays a crucial role in maintaining the bosonic nature of the oscillator.

Fermionic Harmonic Oscillator

As mentioned earlier, Pauli's exclusion principle demands that the fermionic states can be either $|0\rangle$ or $|1\rangle$. Equivalently, the state $|2\rangle$ must be zero. Defining $a_f^\dagger$ and a_f as the fermionic creation and annihilation operators, the number state $|2\rangle$ from (2) will be

$$|2\rangle \propto (a_f^\dagger)^2|0\rangle,$$

which is zero if and only if $(a_f^\dagger)^2 = \frac{1}{2}\{a_f^\dagger, a_f\dagger\} = 0$, where the bracket in parenthesis is called *anticommutator bracket* defined as $\{A, B\} = AB + BA$ for any two operators A, B. Hence, the property of fermions is achieved by the *anticommutation* relations.

Therefore, the restriction imposed by Pauli's exclusion principle results in the following relations for the operators $a_f, a_f^\dagger$ and the fermionic Hamiltonian H_f:

$$\begin{aligned} \{a_f, a_f\} &= \{a_f^\dagger, a_f^\dagger\} = 0 \text{ and } \{a_f, a_f^\dagger\} = 1, && (7) \\ H_f &= \hbar\omega_f\left(a_f^\dagger a_f - \frac{1}{2}\right), && (8) \\ H_f|n_f\rangle &= E_{n_f}|n_f\rangle, \text{where} \\ &\qquad E_{n_f} = (n_f - 1/2)\hbar\omega_f, && (9) \end{aligned}$$

where $n_f = 0, 1$. In fact, the anticommutation relation in (7) has been used to derive the Hamiltonian H_f from (1).

With this background on bosonic and fermionic harmonic oscillators, we shall study the supersymmetric oscillator obtained from them in the next section.

Supersymmetry and Supersymmetric Oscillator

Consider a system which is a combination of one bosonic and one fermionic oscillator. Assume that there is no interaction between the two oscillators, i.e., fermionic operators commute with bosonic operators. The state of the combined system can be represented as $|n_b, n_f\rangle$. With the aim of understanding supersymmetry, we define an operator $Q = a_b a_f^\dagger$ and its conjugate $Q^\dagger = a_f a_b^\dagger$. The action of these operators on any state of the system is

$$\begin{aligned}
Q|n_b, n_f = 0\rangle &= a_b a_f^\dagger |n_b, n_f = 0\rangle = \sqrt{n_b}|n_b - 1, \\
&\qquad n_f = 1\rangle, \\
Q^\dagger|n_b, n_f = 1\rangle &= a_f a_b^\dagger |n_b, n_f = 1\rangle = \sqrt{n_b + 1}|n_b + 1, \\
&\qquad n_f = 0\rangle.
\end{aligned}$$

Note that $a_f^\dagger, a_f$ act only on the fermionic state, leaving the bosonic state untouched, because the two systems are non-interacting. Similarly $a_b^\dagger, a_b$ act only on the bosonic state.

The operator Q changes $n_f = 0$ to $n_f = 1$, and $Q^\dagger$ does the reverse. Given any state containing even number of fermions, it is equivalent to a state containing only bosons. In other words, $|n_b, n_f \equiv \text{even number}\rangle$ is considered to behave like a bosonic state. Similarly, a state with n_f odd and any number of bosons is equivalent to a fermionic state. Therefore, we can say that the operator Q transforms a bosonic state into a fermionic state and the operator $Q^\dagger$ transforms a fermionic state into a bosonic state. Such a transformation performed by the operator Q or $Q^\dagger$ is called *supersymmetric transformation* and Q , $Q^\dagger$ are called the *generators* of the supersymmetric transformation.

We have basically combined a bosonic system and a fermionic system. Therefore, the Hamiltonian for such a system, with no interactions between the fermionic and bosonic oscillators, will be the sum of the individual Hamiltonians, i.e., $H = H_b + H_f$. From here on, we choose the same frequency ω for both the bosonic and fermionic oscillators. Thus

$$H = \omega(a_b^\dagger a_b + a_f^\dagger a_f).$$

Our next step is to determine the symmetry possessed by the combined system. Recalling the formal definition of any symmetry, we shall evaluate the commutator of H and the generators Q, $Q^\dagger$. This will determine whether the combined system is invariant under supersymmetry transformation. Expanding one such commutator, we get

$$[H, Q] = \omega[a_b^\dagger a_b + a_f^\dagger a_f \ , a_b a_f^\dagger] =$$

$$\omega \left(a_b^\dagger a_b a_b a_f^\dagger + a_f^\dagger a_f a_b a_f^\dagger - a_b a_f^\dagger a_b^\dagger a_b - a_b a_f^\dagger a_f^\dagger a_f \right)$$

$$= \omega \left(a_b^\dagger a_b a_b a_f^\dagger + a_b(1 - a_f a_f^\dagger) a_f^\dagger - (a_b^\dagger a_b + 1) a_b a_f^\dagger \right) = 0.$$

Here we have used the commutation and anticommutation relations of the fermionic and bosonic operators, and the non-interacting property of the bosonic and fermionic oscillators, to simplify the bracket, and we see that the Hamiltonian of the combined system H commutes with the generator Q. This crucially depended on choosing identical frequency ω for the bosonic and fermionic oscillators. Similarly, we can also show that $[H, Q^\dagger] = 0$. The zero commutator bracket implies that the combined system described by the Hamiltonian H is invariant under the supersymmetric transformation. Hence, it is appropriate to call such a combined system as *supersymmetric oscillator*.

The action of zero commutator bracket $[H, Q] = 0$ on a state $|n_b, n_f = 0\rangle$ will prove that the states $|n_b, n_f = 0\rangle$ and $Q|n_b, n_f = 0\rangle \equiv |n_b - 1, n_f = 1\rangle$ are *degenerate eigenfunctions* of the Hamiltonian H. It is not difficult to verify that the generators $Q, Q^\dagger$ obey the following relations:

$$\{Q, Q^\dagger\} = H \ , \ \{Q, Q\} = 0 \ , \ \{Q^\dagger, Q^\dagger\} = 0 \qquad (10)$$

Though we have illustrated supersymmetry and its generators using supersymmetric oscillator as an example, we can take *the above relations as the defining equations for any supersymmetric system.* In the next section, we shall present the formalism of obtaining any supersymmetric system from two systems called partner systems.

Supersymmetric Systems

Let us consider any general Hamiltonian H_1, describing a system in a potential $V_1(x)$:

$$H_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x),$$

whose energy eigenvalues and eigenstates are $E_n^{(1)}$ and $\psi_n^{(1)}(x)$, respectively. Using Schrödinger equation, we can rewrite the form of the potential $V_1(x)$ in terms of ground state energy $E_0^{(1)}$ and ground state eigenfunction $\psi_0(x)$ as

$$V_1(x) = \frac{\hbar^2}{2m}\frac{\psi_0^{(1)''}(x)}{\psi_0^{(1)}(x)} + E_0^{(1)}.$$

In order to obtain a supersymmetric system, we need to combine this system with another system which will satisfy all the defining relations (10) of supersymmetry. For this purpose, it will be useful to factorise H_1 in the following way:

$$H_1 = A_1^\dagger A_1 + E_0^{(1)}, \text{where}$$

$$A_1 = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x), A_1^\dagger = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x),$$

where the function $W(x)$ is called the superpotential. The factorised form of the Hamiltonian implies $V_1(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) + E_0^{(1)}$ Extending the role of A_1 and

$A_1^\dagger$ as the creation and annihilation operator for the system described by H_1, we can impose the following condition on the ground state wave function: $A_1\psi_0^{(1)}(x) = 0$. Hence, the solution for $W(x)$ in terms of the ground state wave function is

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\frac{d}{dx}\psi_0^{(1)}(x)}{\psi_0(x)} = -\frac{\hbar}{\sqrt{2m}}\frac{d\ \ln\psi_0^{(1)}(x)}{dx} \quad (11)$$

Now, using $A_1^\dagger$ and A_1, we can construct a new Hamiltonian, H_2, as follows :

$$H_2 = A_1A_1^\dagger + E_0^{(1)} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_2(x),$$

whose energy eigenvalues and eigenfunctions are $E_n^{(2)}$ and $\psi_n^{(2)}(x)$, respectively. The potential $V_2(x)$ in terms of superpotential is

$$V_2(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) + E_0^{(1)}. \quad (12)$$

The two Hamiltonians H_1 and H_2 and their corresponding eigenfunctions, $\psi_n^{(1)}(x)$ and $\psi_n^{(2)}(x)$, can be rewritten in a compact form as follows: Suppose we construct a matrix Hamiltonian

$$H = \begin{bmatrix} H_1 - E_0^{(1)} & 0 \\ 0 & H_2 - E_0^{(1)} \end{bmatrix},$$

whose eigenfunctions are the column vectors $\Psi_n(x) = \begin{pmatrix} \psi_n^{(1)}(x) \\ \psi_n^{(2)}(x) \end{pmatrix}$ It is straightforward to check that the matrix Hamiltonian H commutes with the following matrix operators:

$$Q = \begin{bmatrix} 0 & 0 \\ A_1 & 0 \end{bmatrix} \text{ and } Q^\dagger = \begin{bmatrix} 0 & A_1^\dagger \\ 0 & 0 \end{bmatrix}$$

Further, these matrices obey the defining relations (10) of supersymmetry. Hence, we can say that the combined system described by matrix Hamiltonian H is a *supersymmetric system.*

Applying the zero commutator bracket $[H, Q] = 0$ and $[H, Q^\dagger] = 0$ on $\Psi_n(x)$ gives the following equations:

$$\begin{aligned} H_2\left(A_1\psi_n^{(1)}(x)\right) &= A_1H_1\psi_n^{(1)}(x) = E_n^{(1)}\left(A_1\psi_n^{(1)}(x)\right), \\ H_1\left(A_1^\dagger\psi_n^{(2)}(x)\right) &= A_1^\dagger H_2\psi_n^{(2)}(x) = E_n^{(2)}\left(A_1^\dagger\psi_n^{(2)}(x)\right) \end{aligned}$$

Thus, when $\psi_n^{(1)}(x)$'s are eigenfunctions of H_1, $A_1\psi_n^{(1)}(x)$ are eigenfunctions of H_2. Similarly, when $\psi_n^{(2)}(x)$'s are eigenfunctions of H_2, then $A_1^\dagger\psi_n^{(2)}(x)$ are eigenfunctions of H_1. Using a little algebra, we can show the following relations :

$$E_n^{(2)} = E_{n+1}^{(1)};\ \psi_n^{(2)}(x) = (E_{n+1}^{(1)} - E_0^{(1)})^{-1/2}A\psi_{n+1}^{(1)}(x);$$

$$\psi_{n+1}^{(1)}(x) = (E_n^{(2)} - E_0^{(1)})^{-1/2}A^\dagger\psi_n^{(2)}(x). \qquad (13)$$

Thus, we have exploited supersymmetry to relate the energy eigenfunctions of one Hamiltonian H_1 in terms of the energy eigenfunctions of the partner Hamiltonian H_2. Therefore, H_1 and H_2 are usually called supersymmetric partner Hamiltonians, and their corresponding potentials V_1 and V_2 are called supersymmetric partner potentials.

An Example

The concept of supersymmetric partner potentials immediately suggests the following use: If we have a potential which is difficult to solve analytically, but its partner potential is relatively easy to solve, then by invoking the properties of supersymmetry we immediately have all the energy eigenvalues and eigenfunctions of the unsolvable potential. Let us illustrate the power of supersymmetry through a simple example.

Consider a system described by a potential $V_1(x)$ (infinite potential well):

$$\begin{aligned} V_1(x) &= 0 \quad \text{for } 0 \le x \le L, \\ &= \infty \quad \text{otherwise,} \end{aligned}$$

whose wave functions and energy eigenvalues are known:

$$\psi_n^{(1)}(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{(n+1)\pi}{L}x\right) \text{ and } E_n^{(1)} = \frac{\pi^2(n+1)^2\hbar^2}{2mL^2},$$

where $n = 0, 1, 2,$ .. For algebraic simplicity, let us take $\hbar = 2m = 1$ [2].

[2] The analysis of the problem is identical even when we include $\hbar$ and $2m$.

The superpotential $W(x)$ can be obtained using the above ground state wave function in (11):

$$W = -\frac{1}{\psi_0^{(1)}(x)} \frac{d\psi_0^{(1)}(x)}{dx} = -\frac{\sqrt{\frac{2}{L}}\frac{\pi}{L}\cos\left(\frac{\pi}{L}x\right)}{\sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right)} =$$

$$= -\frac{\pi}{L}\cot\left(\frac{\pi}{L}x\right)$$

Hence, the supersymmetric partner potential $V_2(x)$ from (12) for this example will be

$$\begin{aligned} V_2(x) &= W'(x) + W^2(x) + E_0^{(1)} = \frac{\pi^2}{L^2}\text{cosec}^2(\frac{\pi}{L}x) + \\ &\quad \frac{\pi^2}{L^2}\cot^2(\frac{\pi}{L}x) + \frac{\pi^2}{L^2} \\ &= \frac{2\pi^2}{L^2}\text{cosec}^2(\frac{\pi}{L}x). \end{aligned}$$

Suppose we started with the above potential $V_2(x)$, we would have no clue about finding the energy eigenvalues and energy eigenfunctions. However, we used supersymmetry formalism to derive the potential as a partner potential. Therefore, using(13), the energy eigenvalues and eigenfunctions are related to the energies and wave functions of the particle in a potential well:

$$E_n^{(2)} = E_{n+1}^{(1)} = \frac{\pi^2(n+2)^2}{L^2},$$

$$\psi_n^{(2)} = [E_{n+1}^{(1)} - E_0^{(1)}]^{-1/2} A\psi_{n+1}^{(1)} =$$

$$= \left[\frac{\pi^2(n+2)^2}{L^2} - \frac{\pi^2}{L^2}\right]^{-1/2} \left(\frac{d}{dx} + W\right) \sqrt{\frac{2}{L}} \sin\frac{(n+2)\pi}{L}x$$

$$= \sqrt{\frac{2}{L[(n+2)^2-1]}} \left((n+2)\cos\frac{(n+2)\pi}{L}x - \cot\frac{\pi}{L}x \;\; \sin\frac{(n+2)\pi}{L}x\right)$$

Thus, using the concept of supersymmetry, we have solved a potential which is very difficult to solve by traditional methods.

Hierarchy of Hamiltonians

So far, we have seen that the ground state wave function of H_1 was used to determine the explicit form of the creation and annihilation operators $A_1^\dagger$ and A_1. Using supersymmetry, we derived the partner Hamiltonian H_2 whose eigenfunctions and energies are given by (13). This procedure can be continued to obtain a tower of Hamiltonians tied by supersymmetry relations. For instance, we can refactorise $H_2 = A_2^\dagger A_2 + E_0^{(2)}$, where the forms of the new creation and annihilation operators ($A_2^\dagger$ and A_2) can be deduced from the ground state wave function of H_2. Similar to the procedure of deriving Hamiltonian H_2 from its supersymmetric partner H_1, we can construct another Hamiltonian H_3 from H_2. This process can be continued. It may be noticed from (13) that each newly constructed Hamiltonian will have one fewer energy eigenstate than the previous one. So if H_1 has s energy eigenstates, then we can construct a hierarchy of $(s-1)$ Hamiltonians, all having the same energy spectra, except that the mth Hamiltonian has $s+1-m$ energy eigenstates. This procedure will relate the energy eigenvalues and eigenfunctions of the m-th Hamiltonian H_m in terms of eigenvalues and wave functions of H_1 in the following way:

$$E_n^{(m)} = E_{n+1}^{(m-1)} = \quad = E_{n+m-1}^{(1)},$$

$$\psi_n^{(m)} = (E_{n+m-1}^{(1)} - E_{m-2}^{(1)})^{-1/2} \quad (E_{n+m-1}^{(1)} - E_0^{(1)})^{-1/2} A_{m-1} \quad . A_1 \psi_{n+m-1}^{(1)},$$

and the corresponding potentials are

$$V_m(x) = V_1(x) - 2\frac{d^2}{dx^2}\ln(\psi_0^{(1)} \quad . \psi_0^{(m-1)}).$$

Thus, knowing all the eigenvalues and eigenfunctions of H_1, we immediately know all the energy values and eigenfunctions of the hierarchy of $(s-1)$ Hamiltonians. Thus, a large number of potentials which are unsolvable by traditional methods can be solved by this method.

Conclusion

In this article, we have presented the salient features of symmetry with emphasis on supersymmetry. We have shown that systems with non-trivial potentials can be exactly solved by exploiting the properties of supersymmetry.

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Suggested Reading

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The French "Gazette des mathematiciens" for January 2002 represented I G Petrovskii to its readers as a "great mathematician, known for his parabolic equation" in the paper by J Trouel. An analogous definition for Hadamard would have been "known as the author of Hadamard's lemma which allows the division of a smooth function by its argument", and Hilbert would have been defined as the "creator of Hilbert space", and Riemann as the "inventor of Riemannian metrics".

Petrovskii was one of the founders of the Moscow Mathematical School, the rector of Moscow University during about 20 of its best years and was one of the deepest and most creative mathematicians of the 20th century. The work of Petrovskii on Hilbert's 16th problem laid the foundations for the new subject of real algebraic geometry which continues to develop actively to this day.

V I Arnold in *Russian Mathematical Surveys*, Vol.57, 2002