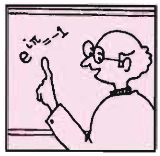
Classroom



In this section of* Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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A Number-theoretic Game

Consider any four natural numbers a, b, c, d and look at the transformation $(a,b,c,d) \mapsto (|a-d|, |b-a|, |c-b|, |d-c|)$.

On performing repeated iterations, what do we observe? For example, $(1,3,7,15) \mapsto (14,2,4,8) \mapsto (6,12,2,4) \mapsto (2,6,10,2) \mapsto (0,4,4,8) \mapsto (8,4,0,4) \mapsto (4,4,4,4) \mapsto (0,0,0,0) \mapsto$

Once the zero sequence is reached, it clearly stays there. Does every sequence lead to the sequence $(0,0,0,0)$? In fact, for any $n \geq 2$, if we start with a sequence of n natural numbers $(a_0, a_1, \quad , a_{n-1})$ and consider the transformation $(a_0, \quad , a_{n-1}) \overset{T}{\mapsto} (|a_0 - a_{n-1}|, |a_1 - a_0|,$ $|a_{n-1} - a_{n-2}|)$, do finitely many repetitions always lead to the sequence $(0, 0, \quad , 0)$ for any starting sequence?

Our aim is to inspect the behaviour of this transformation and find out what n must be in order that every n-tuple to reach the zero sequence.

When $n = 2$, trivially every $(a, b) \mapsto (|a-b|, |a-b|) \mapsto (0, 0)$.

But this is not the case for $n = 3$ or $n = 5$. For instance, $(1,1,0) \mapsto (1,0,1) \mapsto (0,1,1) \mapsto (1,1,0) \mapsto$ being a cycle can never reach the zero tuple. Similarly, six iterations of the tuple $(1,3,4,9,12)$ lead to a tuple consisting of 1's and 0's viz., $(1,1,1,0,1)$ which creates a cycle of length 15.

This leads us to suspect that any tuple leads to a sequence of 0's and 1's. One may perhaps guess that for odd n, there are always tuples never leading to zero and that for all even n, perhaps each tuple does lead to the zero tuple. As we shall see, the first guess is correct while the example $n = 6$ shows that the second one is too optimistic.

The 6-tuple $(1,0,1,0,0,0)$ is easily verified to give rise to a cycle of length 6. We shall prove that it is exactly for the powers of 2 that every tuple leads to the zero tuple and for the other n, we can give cycles which never reach zero. The method of proof will make it clear as to how we produced the above cycles. Let us start with any $n \geq 2$.

We observe:

$T(a_0, \quad , a_{n-1}) \equiv (a_0 + a_{n-1}, a_1 + a_0, \quad , a_{n-1} + a_{n-2})$ mod 2.

This is obvious since $|x - y| \equiv x + y$ mod 2 for any $x, y \in N$. Now, we shall introduce a tool which makes it very convenient to study the effect of the transformation T at least modulo 2.

Let us denote by $F_2[t]$ the polynomials in a variable t with coefficients from the field F_2 of two elements 0 and 1. The usual division algorithm for polynomials holds for these polynomials too. Consider the 'quotient ring' $F_2[t]/(t^n - 1)$ of the polynomial ring in one variable t over F_2; this can simply be thought of as the set of remainders of polynomials when divided by the polynomial $t^n - 1$ in $F_2[t]$. Equivalently, one can think of this as the set of all polynomials $f(t) \in F_2[t]$, where the relation $t^n = 1$

can be used while adding and multiplying polynomials.

Let us associate to a sequence $a = (a_0, \quad , a_n)$ of natural numbers, the polynomial $a(t) = \overline{a_0} + \overline{a_1}t + \overline{a_2}t^2 + \quad + \overline{a_{n-1}}t^{n-1}$ in $F_2[t]/(t^n - 1)$. Here $\overline{a_i} = 0$ or 1 according as whether a_i is even or odd. Note that any two n-tuples a and b whose corresponding entries differ only by even numbers, correspond to the same polynomial.

The following lemma brings out what is special about powers of 2 in our situation.

Lemma:

(a) *Under the above identification, the polynomial* $(1 + t)a$ *corresponds to the sequence* $T(a)$ mod 2.

(b) *If n is a power of 2, then* $(1+t)^n = 0$ *in* $F_2[t]/(t^n-1)$.

(c) *Suppose $n = 2^r d$, where $d > 1$ is odd. Let m be the order of 2 in the multiplicative group $\mathbb{Z}_d^\times$ of integers coprime to d; this simply means that $2^m \equiv 1 (\mathrm{mod}\ d)$ and m is the smallest such natural number.*

Then $(1 + t^{2^r})^{2^m} = 1 + t^{2^r}$ *in* $F_2[t]/(t^n - 1)$.

Proof:

(a) $(1+t)(\overline{a_0}+\overline{a_1}t+\overline{a_2}t^2+. \ .+\overline{a_{n-1}}t^{n-1}) = (\overline{a_0}+\overline{a_{n-1}})+(\overline{a_1}+\overline{a_0})t+\cdot \ \cdot+(\overline{a_{n-1}}+\overline{a_{n-2}})t^{n-1}$ in $F_2[t]/(t^n-1)$. Thus, (a) follows.

(b) $n = 2^r \Rightarrow$ all the binomial coefficients $\binom{n}{d}$; $0 < d < n$ are even. So,

$$(1 + t)^n = 1 + t^n = 1 + 1 = 0 \in F_2[t]/(t^n - 1).$$

(c) Write $2^m = 1 + de$. Note that $m > 1$ as d is odd > 1. Then $(1+t^{2^r})^{2^m} = 1+t^{2^r \cdot 2^m}$ in $F_2[t]$ as above since $\binom{2^m}{\ell}$ is even. This is further equal to $1 + t^{2^r(1+de)} = 1 + t^{2^r} \ t^{ne} = 1 + t^{2^r}$ in $F_2[t]/(t^n - 1)$.

Remark.

As a consequence of the lemma, we see that if n is a power of 2, then for any n-tuple $a = (a_0, \quad, a_{n-1})$ of non-negative integers, $T^n(a)$ is an n-tuple of non-negative, even integers. Here, we have written T^n for T iterated n times.

For $a = (a_0, \quad, a_{n-1}) \in N^n$, let us write $\ell(a) = \max_{0\leq i\leq n-1} [\log_2 a_i]$ and $M(a) = \max_{0\leq i\leq n-1} a_i$. In other words, $2^{l(a)}$ is the highest power of 2, which is bounded by some a_i. Now we can now prove the main result.

Theorem: (a) *Let* $n = 2^r$ *Then,*

$T^{n(\ell(a)+1)}(a) = (0, 0, \quad, 0).$

(b) *Let* $n = 2^r d$ *with d odd* > 1. *Write m for the order of* 2 *in* $\mathbb{Z}_d^\times$ *Consider the sequence* $a = (1, 0, \quad, 0, 1, 0,$ $0)$, *where* 1*'s are at the first and the* $(2^r + 1)$*-th places. Then,* $T^{2^{m+r}-2^r}(a) = a$.

Note that, when n is not a power of 2, (b) shows that the sequence a in (b) never reaches zero for, if it did, the iterations after that would continue to produce the zero sequence.

Proof:

(a) Note first that $M(T(a)) \leq M(a)$, $\forall$ sequences $a \in N^n$. Also, note that $T(2a) = 2T(a)$, $\forall\ a \in N^n$. Now, by the above remark, $T^{n(\ell(a)+1)}(a) = 2^{\ell(a)+1}c$ for some $c \in N^n$.

So, $2^{\ell(a)+1} \quad M(c) \leq M(a)$. Since $2^{l(a)}$ is the highest power of 2 which is less than or equal to some a_i, this would lead to the absurd statement $\ell(a)+1 \leq \ell(a)$ unless $M(c) = 0$. Therefore, $M(c) = 0$ i.e., $c = (0, 0, \quad, 0)$.

(b) Consider $a = (1, 0, \quad, 0, 1, 0, \quad, 0)$ where 1's are at the first and $(2^r + 1)$-th place. Here $n = 2^r d$ with d odd > 1. Under the correspondence with $F_2[t]/(t^n - 1)$, the above element a corresponds modulo 2 to $1+t^{2^r}$ Once again, the lemma gives $T^N(a) \equiv (1+t)^N a \bmod 2,\ \forall\ N$. Taking $N = 2^{m+r} - 2^r$ and using the lemma, we get $T^{2^{m+r}-2^r}(a)$ corresponds in $F_2[t]/(t^n-1)$ with $(1+t)^{2^{m+r}-2^r}$ $(1+t^{2^r}) \equiv (1+t)^{2^{m+r}-2^r}(1+t)^{2^r} \equiv (1+t)^{2^{m+r}} \equiv (1+t^{2^r})^{2^m} \equiv 1+t^{2^r}$ But 'a' itself corresponds with $1+t^{2^r}$ in $F_2[t]/(t^n-1)$. So, $T^{2^{m+r}-2^r}(a) \equiv a \bmod 2$. In other words, $T^{2^{m+r}-2^r}(a) \equiv (a_0+2b_0, a_1+2b_1, \quad, a_{n-1}+2b_{n-1})$ for some $(b_0, \quad, b_{n-1}) \in \mathbb{Z}^n$.

Since a has only 0's and 1's as entries, $T^N(a)$ also has only 0's and 1's as entries $\forall\ N$. In particular, $(a_0 + 2b_0, \quad, a_{n-1} + 2b_{n-1})$ has only 0's and 1's. This forces $b_0 = b_1 = \quad = b_{n-1} = 0$. Thus $T^{2^{m+r}-2^r}(a) = a$. The theorem is proved.

Please Note

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Direct observation of neutrino oscillations at the Sudbury Neutrino Observatory by B Ananthanarayan and Ritesh K Singh

The published results of SNO on the evidence for neutrino oscillations relies on the observation of the gamma radiation from neutron capture on deuterium, and not from data taken after salt addition, as incorrectly stated in our article. The radiochemical measurements used tetrachloroethylene (C_2Cl_4, also called perchloroethylene), not carbon tetrachloride (CCl_4) as stated. It may also be noted that the ES reactions also involve NC and not just CC events. We thank the SNO collaboration and M V N Murthy for pointing out these errors.

Resonance, Vol.6, June 2001, The poster of 'Isomers of Benzene'

The correct IUPAC name of prismane should be Tetracyclo [2.2.0.0 ^{2,6}, $0^{3,5}$] hexane instead of [3.1.0.0 ^{2,4}, $0^{3,6}$].