S Minakshisundaram. A Glimpse into his Life and Work

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During the period 1900-1950 India witnessed the emergence of several pure mathematicians such as K Ananda Rau (1893-1966), R Vaidyanathaswamy (1894-1960), T Vijayaraghavan (1902-1955), S S Pillai (1901-1950) and S Minakshisundaram (1913-1968). With the exception of Vaidyanathaswamy who studied logic, set theory and general topology, all the rest were first class analysts. This is not at all surprising given the fact that both Ananda Rau and Vijayaraghavan were students of G H Hardy and the other two studied with Ananda Rau. In this article we would like to introduce Minakshisundaram to the readers of *Resonance* and give a brief summary of his work. For more details about his life and personality we refer the readers to the obituary written by K G Ramanathan [8]. Indeed, we have drawn generously from this article.

Subbaramiah Minakshisundaram was born in Trichur, Kerala on October 12, 1913. His father was originally from Salem, Tamilnadu (not far away from the birth place of Ramanujan) and so Minakshisundaram had all his education in Madras. He took his BA (Hons.) in Mathematics from Loyola College in 1934 securing first class in the Madras University examination. Though there was always a strong tradition of scholarship and learning in and around Madras, many brilliant young men of that time used to opt for the more lucrative and prestigious administrative services. But young Minakshisundaram was different – he joined Madras University as a research scholar and started working with Ananda Rau.

Keywrods
S Minakshisundaram, V K Patodi.

After taking the DSc degree from Madras University in 1940, Minakshisundaram found himself without a job.

Thanks to the timely help of Fr. Racine who was professor at Loyola College, he could earn a living by coaching students for the university examinations. During these years he and Fr. Racine organised a weekly Mathematics Seminar which attracted many enthusiastic participants like K Chandrasekharan and K G Ramanathan. Fortunately, he got the job of a lecturer at Andhra University in Waltair.

In 1944, Marshall H Stone was in Madras and he wanted to meet the best young mathematicians there, especially Minakshisundaram and Chandrasekharan. M H Stone, who was a Professor at Chicago, had a reputation for 'discovering' young talents and shaping their career. For both Minakshisundaram and Chandrasekharan this meeting with Stone was a turning point. By the efforts of Stone they were offered a membership at the Institute for Advanced Study in Princeton. The atmosphere at the Institute gave him a great boost and it was there where Minakshisundaram's best mathematical works were done. He collaborated with the Swedish mathematician Åke Pleijel and wrote his most quoted paper.

Minakshisundaram returned to India in 1948 by which time he was internationally recognised for his brilliant work. Early in 1950 Andhra University promoted him to full professorship in Mathematical Physics. Soon after he became a professor, he spent a few months at the Tata Institute of Fundamental Research, Bombay where he collaborated with K Chandrasekharan in writing of the monograph 'Typical Means'. (K Chandrasekharan was brought to India in 1949 by Homi Bhabha to build the School of Mathematics in TIFR which he did with great success). Minakshisundaram made a couple of brief visits to the US and in 1958 went to Edinburg to give a half-hour lecture on Hilbert algebras at the International Congress of Mathematicians.

Minakshisundaram collaborated with Åke Pleijel, K Chandrasekharan, O Szasz and C T Rajagopal. He also had several research students working with him but unfortunately most of them gave up research after taking their PhD. He tried hard to generate enthusiasm in his students and colleagues for mathematical research. But he found the atmosphere in an Indian University stifling for creative work. He longed for a place like Princeton where he could work without any hindrance. He was appointed a Professor at the newly created Institute for Advanced Studies in Simla but by then his health was deteriorating after a bad heart attack. He passed away on August 13, 1968.

We now proceed to describe some aspects of Minakshisundaram's work. As this article is meant for *Resonance* readers, we are not in a position to go beyond an outline of his various results. Interested readers can go through the papers mentioned under Suggested Reading. A complete list of papers of Minakshisundaram is given at the end of [8].

Minakshisundaram started his mathematical career by working on Tauberian theorems and summability results of classical Fourier analysis. Then under the influence of M R Siddiqui of Osmania University, Hyderabad he studied non-linear equations of parabolic and hyperbolic type. His work on partial differential equations formed part of his doctoral dissertation entitled 'Fourier Ansatz and non-linear parabolic equations' An important outcome of his thesis is a long series of papers on eigenfunction expansions associated to boundary value problems. He then investigated the associated Zeta function using heat kernel culminating in his famous paper with Åke Pleijel [5].

In order to appreciate the results of Minakshisundaram on non-linear parabolic equations let us begin by considering the linear equation

$$\frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = 0, \quad u(x,0) = f(x) \tag{1}$$

with the boundary condition $u(0,t) = u(\pi,t) = 0$ for all $t \geq 0$. Note that for each natural number n the function $u_n(x,t) = e^{-n^2t}\sin nx$ satisfies the equation with $u_n(x,0) = \sin nx$. If the initial condition $f(x)$ can be expanded as

$$f(x) = \sum_{n=1}^{\infty} c_n \sin nx \tag{2}$$

then a formal solution of (1) is given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2t} \sin nx. \tag{3}$$

This method is well-known and goes back to D Bernoulli (for the case of the wave equation) around 1750. In 1807, Joseph Fourier asserted that any f with $f(0) = f(\pi) = 0$ can be expanded as in (2) and gave the formula (3) for the solution of (1). With this investigation mathematics saw the birth of Fourier series which led to the development of analysis.

Since the work of Minakshisundaram depends heavily on 'heat kernels' let us spend some more time on the above equation (1). The constants c_n appearing in (2) are given by Fourier's formula

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx \tag{4}$$

and hence the formal solution u in (3) can be written as

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n^2t}\left(\int_0^{\pi} f(y)\sin ny \, dy\right)\sin nx. \tag{5}$$

Defining a function $K(x, y, t),\ t > 0$ by the equation

$$K(x, y, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny$$

we can represent the solution u as

$$u(x, t) = \int_0^\pi K(x, y, t)\ f(y)\ dy. \tag{6}$$

The function $K(x, y, t)$ is called the heat kernel for the operator $\frac{d^2}{dx^2}$.

Consider now the inhomogeneous equation

$$\frac{\partial}{\partial t} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) = g(x, t). \tag{7}$$

A solution of this problem is given by

$$u(x, t) = \int_0^\pi \int_0^t K(x, y, t - s) g(y, s) ds\ dy \tag{8}$$

as can be easily checked. But things become more and more complicated if g also depends on u and u_x. In his work, Minakshisundaram addressed the problem

$$\frac{\partial}{\partial t} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) = g(x, t, u, u_x). \tag{9}$$

Assuming that g, as a function of x, can be expanded in terms of $\sin nx$, Minakshisundaram looked for a solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin nx. \tag{10}$$

This leads to an infinite system of nonlinear integral equations which Minakshisundaram solved by appealing to a fixed point theorem of Schauder thus proving the existence and in some cases uniqueness of solutions of nonlinear equations of parabolic type.

The methods used in his thesis gave rise to a new kind of summability, which he called the Bessel summability. A detailed study of this was taken up by K Chandrasekharan in 1942 (then an MSc student) and much later by M S Rangachari (1965). Another important outcome is his work on generalised Fourier expansions which we describe now. In solving the heat equation (1) we have made use of the functions $\varphi_n(x) = \sin nx$ which are eigenfunctions of the boundary value problem

$$\frac{d^2}{dx^2}u(x) = \lambda u(x), \quad u(0) = u(\pi) = 0. \tag{11}$$

Consider an analogue of this problem in two dimensions. Suppose Ω is a bounded connected open subset of $I\!R^2$ with a smooth boundary $\partial\Omega$. Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ be the Laplacian and consider the Dirichlet problem

$$\Delta u = -\lambda^2 u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{12}$$

By spectral theory of self adjoint operators there exists a sequence $\lambda_n \geq 0$, $\lambda_n \to \infty$ of eigenvalues with corresponding eigenfunctions $\varphi_n(x)$, $x = (x_1, x_2)$. That is, we have

$$\Delta\varphi_n(x) = -\lambda_n^2\varphi_n(x), \; x \in \Omega, \; \varphi_n(x) = 0, \; x \in \partial\Omega. \tag{13}$$

The eigenfunctions φ_n can be normalised so that

$$\int_\Omega |\varphi_n(x)|^2 dx = 1.$$

Given such a sequence of eigenfunctions φ_n associated to the boundary value problem (12) one is tempted to solve the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 \text{ in } \Omega, \quad u(x,t) = 0 \text{ on } \partial\Omega \tag{14}$$

for $t > 0$ with the initial condition $u(x, 0) = f(x)$ by assuming a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n \, e^{-\lambda_n^2 t}\varphi_n(x). \tag{15}$$

This is clearly a solution of (14) but the initial condition will be satisfied only if we have an expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \, \varphi_n(x). \tag{16}$$

Thus we are naturally led to the eigenfunction expansion (16) which is sometimes called generalised Fourier expansion of f.

Now we can ask questions similar to those one asks concerning classical Fourier series. The above expansion (16) with

$$c_n = \int_{\Omega} f(x)\bar{\varphi}_n(x)dx$$

converges in $L^2(\Omega)$ whenever $f \in L^2(\Omega)$ but for other modes of convergence (eg. pointwise) one has to assume further conditions on f or appeal to some summability methods (eg. Riesz means). This is the case even with the classical Fourier series. In a series of papers Minakshisundaram proved: (i) uniqueness theorems for the above expansions (ii) Riesz summability and (iii) uniform convergence for a class of functions.

Apart from studying generalised Fourier expansions, Minakshisundaram also proved several interesting results on classical multiple Fourier series. In a joint work with K Chandrasekharan he studied the (Bochner-) Riesz means associated to double Fourier series. In an earlier work with O Szász he studied the absolute convergence of multiple Fourier series. As remarked by S Bochner in his reviews of these papers, the conditions on the function involve properties of its spherical means. This is quite remarkable since the interplay between the regularity of spherical means and pointwise convergence of Fourier series has been studied recently by M Pinsky and others.

Returning to the eigenfunctions $\varphi_n(x)$ associated to the boundary value problem (12) consider the following 'Zeta function'

$$Z(s) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n^{2s}}. \qquad (17)$$

In a series of papers Minakshisundaram started investigating properties of this function. His aim was to determine the asymptotic distribution of the eigenvalues λ_n^2 and the eigenfunctions φ_n. H Weyl, T Carleman and ÅPleijel had done some pioneering work on these problems. The novelty of the approach taken by Minakshisundaram lies in the fact that he used the fundamental solution of the heat equation given by

$$K(x,y,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t}\varphi_n(x)\varphi_n(y) \qquad (18)$$

in studying these problems. Using properties of the heat kernel $K(x,y,t)$ he was able to generalise the asymptotic results of Carleman.

The Swedish mathematician ÅPleijel (who, incidentally, was son-in-law of M Riesz) was visiting the Institute at the same time as Minakshisundaram. They started a fruitful collaboration which resulted in a very influential paper [5]. In this, they studied the Zeta function associated to the Laplace-Beltrami operator Δ on a compact Riemannian manifold without boundary. Using heat kernel methods they were able to prove that the series

$$Z(s) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n^{2s}}$$

(where now $\Delta\varphi_n = -\lambda_n^2\varphi_n$ on M) can be analytically continued to the whole s-plane as an entire function of s with zeros at non-positive integers when $x \neq y$ and as a meromorphic function when $x = y$. This led to the asymptotic behaviour

$$\sum_{\lambda_n^2<\lambda} \varphi_n(x)^2 \sim \frac{\lambda^{\frac{d}{2}}}{(2\sqrt{\pi})^d\,\Gamma(\frac{d}{2}+1)},$$

where d is the dimension of M. Integrating over M one obtains for $N(\lambda)$, the number of eigenvalues less than λ, the asymptotic behaviour

$$N(\lambda) \sim \frac{c_0 \lambda^{\frac{d}{2}}}{(2\sqrt{\pi})^d \, \Gamma(\frac{d}{2}+1)},$$

where c_0 is the volume of M.

The ideas used in this paper turned out to be very useful in studying geometric properties of manifolds using analytic techniques. An interesting problem is to investigate how the eigenvalues λ_n^2 reflect the geometry of M. In the language of M Kac, 'Can we hear the shape of a drum?' That is, thinking of M as a vibrating membrane and λ_n^2 as its fundamental tones, is it possible to figure out the shape of the drum by simply listening to the music it produces? In 1964, Milnor showed that the eigenvalues cannot distinguish isometric manifolds.

The idea of Minakshisundaram and Pleijel was to look at the heat kernel

$$K(x,y,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \varphi_n(x) \, \varphi_n(y) \tag{19}$$

along the diagonal when t is small. For the Fourier series, that is for the manifold S^1 (the unit circle in $\mathbb{R}^2$),

$$K(x,x,t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-n^2 t}$$

which is a theta function, usually denoted by $\theta(t)$. In view of Jacobi's identity we have

$$K(x,x,t) = (4\pi t)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2 n^2}{t}}$$

from which we obtain the asymptotic expansion

$$K(x,x,t) \sim (4\pi t)^{-\frac{1}{2}} \{1 + 0(1)\}$$

for small t. Minakshisundaram and Pleijel established such an expansion in the general case. For small t they showed that the kernel defined in (19) satisfies

$$K(x,x,t) \sim (4\pi t)^{-\frac{d}{2}}\{1 + k_1(x)t + k_2(x)t^2 + \quad .\}. \quad (20)$$

The coefficients $k_j(x)$ are now called Minakshisundaram coefficients.

The computation and geometric interpretation of Minakshisundaram coefficients is still an open problem. When $d = 2$, Mc Kean and Singer showed that k_1 and k_2 are related to the curvature properties of the manifold. Integrating (20) over M we get

$$\sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sim (4\pi t)^{-\frac{d}{2}} c_0\{1 + a_1 t + a_2 t^2 + \quad .\} \quad (21)$$

which shows that the dimension d and the volume c_0 of the manifold can be 'heard' from the spectrum. For the action of the Laplace–Beltrami operator on p forms, $0 \leq p \leq d$ one has a similar expansion

$$\sum_{n=1}^{\infty} e^{-\lambda_{n,p}^2 t} \sim (4\pi t)^{-\frac{d}{2}} c_0 \left\{ \binom{d}{p} + a_{1,p} t + a_{2,p} t^2 + \quad \right\}$$

In a 1970 paper V K Patodi of TIFR computed the coefficients $a_{1,p}$ and $a_{2,p}$ for all p. Thus the idea of using heat kernel has proved to be very fruitful and extremely powerful. It has led to the work of Atiyah, Bott and Patodi on a new approach to the index theorem for elliptic operators.

Suggested Reading

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