Arithmetic Nature of Sums of Certain Convergent Series

S D Adhikari and N Saradha

1. Introduction

The study of infinite series has been a scintillating subject and its connection with the theory of numbers has a very long history. In our college days, we are taught various tests for deciding whether a given series will converge or not. When the series is convergent, one is interested in finding the sum of the series and the arithmetic nature of the sum. This still remains a challenging question for many well-known convergent series. In the present article, we shall discuss the questions of irrationality and transcendence of sums of some infinite series.

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2. Definitions and Some Basic Observations

A real number α is called *algebraic* if there exists a polynomial $P(x) = a_0 + a_1x + \quad + a_nx^n$, where a_i's are integers, not all zero, such that $P(\alpha) = 0$. A real number is called *transcendental* if it is not algebraic. These definitions of algebraic and transcendental numbers hold for complex numbers as well.

Among the non-zero polynomials with integral coefficients satisfied by an algebraic number α, there exists a polynomial of least degree which is unique upto a constant. One such polynomial with the leading coefficient positive and having 1 as the gcd (greatest common divisor) of its coefficients, is called *the minimal polynomial* of α. We observe that the minimal polynomial is irreducible over the field of rational numbers. The degree of the minimal polynomial of α is called the *degree* of the algebraic number α and the maximum of the absolute values of the coefficients of its minimal polynomial is called the *height* of α. Thus, if $\alpha = \frac{a}{b}$ is a rational num-

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ber, where a and b (> 0) are integers with gcd $(a, b) = 1$, then the minimal polynomial of α is $bx - a$ and hence the degree of α is one and its height is $\max(|a|, |b|)$. If $\alpha = \sqrt{2}$, then the minimal polynomial is $X^2 - 2$ and hence the degree as well as the height of $\sqrt{2}$ is 2.

The concept of countability was introduced by Cantor in the year 1874. By his famous diagonal argument, Cantor showed that the set of real numbers is uncountable. It is easy to see that there are only finitely many polynomials with integral coefficients having a fixed height and a fixed degree. Thus the set of algebraic numbers is countable. Therefore, in the set of real numbers, the transcendental numbers are in abundance. However, it is a difficult problem to decide whether a given number (often arising as a geometric constant or through some limiting procedure) is transcendental or not. The irrationality of

$$e = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} +$$

was established by Euler way back in 1744, while the transcendence of e was established by Hermite in 1873. Following the idea of Hermite, in 1882 Lindemann established the transcendence of π which is known to have the series expansion given by

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$$

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See Dutta [1] for a history on the above series expansion of π and its approximate values. In 1844, Liouville established a criterion for a number to be algebraic. This enabled him to construct transcendental numbers by means of infinite series. We shall state and prove Liouville's result in the next section.

One of the most important series in number theory is the series

$$\sum_{n=1}^{\infty} n^{-s},$$

where s denotes a complex number. It can be seen easily that this series converges for Re $s > 1$ and at $s = 1$ it diverges. We denote the sum of the series by $\zeta(s)$. This is known as the Riemann zeta function and is extremely important in the theory of primes. It is known that $\zeta(s)$ is a holomorphic function in the halfplane Re $s > 1$ and it can be continued analytically to a meromorphic function on the whole plane. We are interested in the value of the series at $s = 2, 3, 4,$ It is known that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In fact it is known that $\zeta(2n)$ is a rational multiple of π^{2n} for all integers $n > 0$. We have

$$\zeta(4) = \frac{\pi^4}{90}$$

and in general

$$\zeta(2n) = \frac{2^{2n-1} B_n}{(2n)!} \pi^{2n},$$

where B_n is a rational number known as the n-th Bernoulli number given by the coefficients in the expansion

$$\frac{x}{e^x - 1} = 1 - \frac{1}{2}x + \frac{B_1}{2!}x^2 - \frac{B_2}{4!}x^4 + \frac{B_3}{6!}x^6 -$$

Thus it is clear that the values of $\zeta(s)$ at even integers are transcendental. The transcendence of the values of $\zeta(s)$ at odd integers still remains elusive. There had been some progress towards the corresponding irrationality questions. The first breakthrough in this direction came in 1979 when Apéry [2] established the irrationality of $\zeta(3)$. His proof was long and wearisome. It was simplified beautifully by Beukers (see [3, 4, 5]). The next progress was made by Ball and Rivoal [6] (see also [7]) recently. They showed that the dimension of the vector space generated by $1, \zeta(3), \zeta(5), \quad, \zeta(a)$ over

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the field of rationals is $\geq \frac{1}{3}\log(a)$. Hence there are infinitely many odd positive integers m, for which $\zeta(m)$ is irrational. The irrationality of a single number $\zeta(m)$ for $m > 3$ still remains unsolved. Though one of Rivoal's results [8] says that at least one of the nine numbers $\zeta(5), \zeta(7), \ldots, \zeta(21)$ is irrational and very recently the method has been used to yield further improvements, experts feel that the method will not lead to an answer to the question of irrationality of a particular $\zeta(m)$, where $m > 3$ is an odd integer.

3. Diophantine Approximation and Transcendence

In this section we state and prove Liouville's theorem. The rational numbers are dense in the real number system. Hence given a real number we can always find rationals as near to the real number as we please. Thus given a real number θ and a small positive number ϵ, one asks how large the denominator of a rational number $\frac{p}{q}$ must be to have an approximation

$$|\frac{p}{q} - \theta| \leq \epsilon \tag{1}$$

Let us take $\epsilon = \frac{1}{q^2}$. Then (1) becomes

$$|\frac{p}{q} - \theta| \leq \frac{1}{q^2}. \tag{2}$$

Suppose θ is a rational number, say $\theta = \frac{a}{b}$. Then

$$|\frac{p}{q} - \theta| = |\frac{p}{q} - \frac{a}{b}| = |\frac{bp - aq}{bq}| \geq \frac{1}{bq}.$$

Hence the inequality (2) implies that $q \leq b$. Thus if θ is a rational number, there are only a finite number of solutions of (2) in integers p and q.

Suppose θ is irrational. From the theory of continued fractions, we know that every convergent $\frac{p_n}{q_n}$ of θ satisfies

$$|\frac{p_n}{q_n} - \theta| < \frac{1}{q_n^2}. \tag{3}$$

Therefore every $\frac{p_n}{q_n}$ is a solution of (2) and hence there are infinitely many solutions to the inequality (2) in this case.

On the other hand, when θ is algebraic, we shall see below (Theorem 1) that in the inequality (1), we cannot take ϵ as $\frac{1}{q^h}$ with h as large as we like. It is this property of algebraic numbers which enabled Liouville to construct transcendental numbers.

Liouville's construction of transcendental numbers is based on the fact that transcendental numbes can be approximated better by rational numbers than algebraic, irrational numbers are.

Theorem 1. **(Liouville)** *Let α be a real algebraic number of degree d (≥ 2). Then there exists a constant $c = c(\alpha)$ depending only on α such that for all integers p, q with $q > 0$, we have*

$$|\alpha - \frac{p}{q}| > \frac{c}{q^d}.$$

We see that if $\alpha = \beta + i\gamma$ is a complex number where β and $\gamma (\neq 0)$ are reals, then for integers p, q with $q > 0$, one has

$$|\beta + i\gamma - \frac{p}{q}| \geq |\gamma| \geq \frac{|\gamma|}{q^d}.$$

Thus Theorem 1 is true when the algebraic number α is not real.

Proof of Theorem 1. We may assume that

$$|\alpha - \frac{p}{q}| \leq 1$$

since otherwise the theorem is true by taking, say $c = 1$. Let

$$f(T) = a_0T^d + a_1T^{d-1} + \quad + a_d$$

be the minimal polynomial of α and

$$c_1 = \sup\{f'(T) : \alpha - 1 \leq t \leq \alpha + 1\}.$$

We observe that $f(\alpha) = 0$. Also, $f(\frac{p}{q}) \neq 0$ since f is irreducible over the field of rationals and hence $q^d f(\frac{p}{q})$

is a non-zero integer. Therefore,

$$1 \leq q^d|f(\frac{p}{q})| = q^d|f(\frac{p}{q}) - f(\alpha)| \leq c_1 q^d|\frac{p}{q} - \alpha|. \quad (4)$$

Thus

$$|\frac{p}{q} - \alpha| \geq \frac{1}{c_1 q^d}.$$

This completes the proof of the theorem.

We now see how one can use Theorem 1 to prove that sums of certain infinite series are transcendental. Let

$$\xi = .110001000... = \sum_{n=1}^{\infty} 10^{-n!}$$

and

$$p_j = 10^{j!} \sum_{n=1}^{j} 10^{-n!} \quad q_j = 10^{j!},$$

for integers $j \geq 1$. Then,

$$\begin{aligned} |\xi - \frac{p_j}{q_j}| &= \sum_{n=j+1}^{\infty} 10^{-n!} \\ &< 10^{-(j+1)!} \sum_{n=0}^{\infty} 10^{-n} \\ &= \frac{10}{9}(q_j)^{-(j+1)} \\ &< (q_j)^{-j}. \end{aligned}$$

Since this is true for every integer $j \geq 1$, by Theorem 1 we find that ξ is not an algebraic number. One can take any real number $t > 1$ in place of 10 and by the above argument we find that the number $\sum_{n=1}^{\infty} t^{-n!}$ is transcendental. In fact, several such series with many variations can be constructed. The essential point here is that any number which can be approximated very well by a rapid sequence of rational numbers must necessarily be transcendental.

Suppose α is a quadratic irrational, like $\sqrt{2}$, $\frac{1}{2}(\sqrt{5}-1)$ etc. Then by Theorem 1 and (3) it is clear that in (1) we cannot take ϵ to be less than $\frac{1}{q^{2+h}}$ with $q^h > c^{-1}$. Suppose α is an algebraic number of degree > 2. Then one wishes to improve the inequality in Theorem 1 i.e., one would like to replace $\frac{1}{q^d}$ by $\frac{1}{q^r}$ with some positive number $r < d$. We observe that $r = 2 + \delta$ with $\delta > 0$ is the best possible by (3). In 1909, Thue, a Norwegian mathematician made the first and remarkable progress in this direction. He also established the connection between such diophantine approximations and finiteness of the number of solutions of certain diophantine equations. He showed that d can be replaced by a number $\kappa > \frac{d}{2} + 1$. The next improvement came in 1921 by Siegel who showed κ has to satisfy $\kappa > 2\sqrt{d}$. In 1947, Gelfond and Dyson independently showed that d can be replaced by any κ satisfying $\kappa > \sqrt{2d}$. Finally in 1955, Roth showed that κ with $\kappa > 2 + \epsilon'$ for any $\epsilon' > 0$ does the job. We refer to [18] for a proof of Thue's result. For the other important results mentioned above, one may look into [12] and [13]. We refer to [10] for more facts on approximation to real numbers.

Dirichlet series constitute one of the most useful tools in analytic number theory. In 1837, Dirichlet proved his celebrated theorem that there are infinitely many primes in any arithmetic progression.

4. Infinite Series $\sum_{n=1}^{\infty} \frac{f(n)}{n}$ with f Periodic mod q

The Riemann zeta function we came across in Section 2 is a particular case of a wider class of functions known as Dirichlet series defined as

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where a_n' s are complex numbers. Let us specialise a_n to be $f(n)$, where $f(n)$ is an arithmetical function. These are called Dirichlet series with coefficients $f(n)$. They constitute one of the most useful tools in analytic number theory. In 1837, Dirichlet proved his celebrated theorem that there are infinitely many primes in any arithmetic progression. Suppose we take an arithmetic progression with common difference q. Then Dirichlet analysed the series,

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ known as a Dirichlet character is a complex valued, completely multiplicative arithmetic function which is periodic with period q and satisfying $\chi(n) = 0$ if $\gcd(n,q) > 1$. The character χ_0 defined by

$$\chi_0(n) = \begin{cases} 1 & \text{if } \gcd(n,q) = 1 \\ 0 & \text{if } \gcd(n,q) > 1. \end{cases}$$

is called the principal character. A major step in the proof of Dirichlet's theorem is in showing that

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n} \neq 0,$$

whenever $\chi \neq \chi_0$.

A deep theorem of Baker in the theory of linear forms in logarithms states [12] that

Theorem A.(Baker) *If $\alpha_1, \quad, \alpha_n$ are non-zero algebraic numbers, $\beta_1, \quad, \beta_n$ are algebraic numbers and*

$$\Lambda = \beta_1 \log \alpha_1 + \quad + \beta_n \log \alpha_n,$$

then $\Lambda = 0$ or Λ is transcendental.

It is well known that $L(1,\chi)$ can be expressed as a linear form in logarithms with algebraic coefficients. Hence it follows from Theorem A that

$$L(1,\chi) \text{ is transcendental for } \chi \neq \chi_0.$$

In 1969 Chowla posed the question:

Suppose f is a rational valued periodic function mod q where q is a prime number. Then

$$\text{is } S = \sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

In 1973, Baker, Birch and Wirsing [13] answered this question in the affirmative. Chowla himself gave a proof independently. We shall suppose that the series converges. As an analogue of the transcendence of $L(1,\chi)$ it was observed in [15] that

$$\mathcal{S} \text{ is transcendental.}$$

For the above observation, it is clear from Theorem A that we need only to express $\mathcal{S}$ as a linear form in logarithms of algebraic numbers with algebraic coefficients. Let $\xi = e^{2\pi i/q}$. Then

$$1 + \xi + \quad + \xi^{q-1} = \frac{1-\xi^q}{1-\xi} = 0.$$

In fact, for any integer r we have

$$\sum_{i=1}^{q} \xi^{ri} = \begin{cases} 0 & \text{if } r \not\equiv 0 \pmod q \\ q & \text{if } r \equiv 0 \pmod q. \end{cases} \tag{5}$$

This property will be used to show

$$\sum_{i=1}^{\infty} \frac{f(n)}{n} = -\frac{1}{q} \sum_{s=1}^{q-1} \sum_{r=1}^{q} f(r) \xi^{-rs} \log(1-\xi^s). \tag{6}$$

We follow the argument of Baker, Birch and Wirsing. We also refer to Lehmer [14] for another proof. See also [15]. A necessary and sufficient condition for $\mathcal{S}$ to converge is that

$$\sum_{i=1}^{q} f(i) = 0. \tag{7}$$

We put

$$g(s) = \frac{1}{q} \sum_{i=1}^{q} f(i) \xi^{-is}$$

We note that $g(0) = 0$ by (7). Also, g is periodic mod q.

Now,

$$\begin{aligned}\sum_{s=1}^{q-1} g(s)\xi^{rs} &= \sum_{s=1}^{q} g(s)\xi^{rs} \\ &= \sum_{s=1}^{q} \frac{1}{q} \sum_{i=1}^{q} f(i)\xi^{(r-i)s} \\ &= f(r)\end{aligned}$$

and by (5) the right hand side in (6) is

$$\begin{aligned}-\sum_{s=1}^{q-1} g(s)\log(1-\xi^s) &= \sum_{s=1}^{q-1} g(s) \sum_{h=1}^{\infty} \frac{\xi^{hs}}{h} \\ &= \sum_{h=1}^{\infty} \frac{1}{h} \sum_{s=1}^{q-1} g(s)\xi^{hs} \\ &= \sum_{h=1}^{\infty} \frac{f(h)}{h},\end{aligned}$$

by the equality established above. This completes the proof of (6). We hope that the reader will justify the rearranging of the terms in the infinite series occuring in the above proof.

Let us consider the series

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)}.$$

We have

$$\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)} = \frac{1}{1.2} + \frac{1}{4.5} + \quad =$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} +$$

The series can be written as $\sum_{n=1}^{\infty} \frac{f(n)}{n}$, where

$$f(n) = \begin{cases} 1, & \text{if } n \equiv 1 \pmod 3 \\ -1, & \text{if } n \equiv -1 \pmod 3 \\ 0, & \text{if } n \equiv 0 \pmod 3). \end{cases}$$

Now $\sum_{n=0}^{\infty} \frac{1}{(3n+1)(3n+2)}$ can never be zero, since it is a sum of positive terms. Hence this sum is transcendental. In fact, transcendence of many convergent series of the form $\sum_{n=0}^{\infty} \frac{1}{(qn+r_1)\cdots(qn+r_\mu)}$ with distinct r_i's satisfying $0 < r_i \leq q$ for all i, $1 \leq i \leq \mu$, can be obtained by similar arguments. We refer to [15] and [14] for many more examples.

Let us consider the Fibonacci sequence

$$F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5,$$

It can be seen that

$$F_n = \frac{1}{\sqrt{5}}\{(\frac{1}{2} + \frac{\sqrt{5}}{2})^n - (\frac{1}{2} - \frac{\sqrt{5}}{2})^n\}.$$

By the method explained above it has been shown in [15] that

$$\sum_{n=1}^{\infty} \frac{F_n}{n2^n}$$

is transcendental. On the other hand, it is well known that $\sum_{n=1}^{\infty} \frac{F_n}{2^n} = 2$. Several other series involving Fibonacci numbers were studied in literature by developing a method of Mahler which is different from the method used in the previous paragraph. For instance, it has been shown in [16] that

$$\sum_{k=0}^{\infty} \frac{1}{F_{2^k+1}}$$

is transcendental. We refer to [17] for a detailed study on the method of Mahler.

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