Timoshenko–Wagner–Kappus Torsion Bending Theory and Wind Tunnel Balance Design

S P Govinda Raju and K R Y Simha

S P Govinda Raju is a Professor at the Aerospace Engineering Department, Indian Institute of Science, Bangalore. His interests are fluid mechanics, aircraft design, testing and analysis.

K R Y Simha is a Professor at the Mechancial Engineering Department, Indian Institute of Science, Bangalore. His interests are solid mechanics, fracture mechanics, shock and impact mechanics.

Timoshenko was born in Ukraine on December 23, 1878. After a pleasant primary and secondary education, he studied for five years (1896-1901) at the St. Petersburg Institute of Engineers on ways of communication. Summer tours to Europe during this period inspired him to study under two outstanding German professors: Foppl and Prandtl.

The study of torsion and bending has always been a favourite pursuit in the long history of strength of materials and the theory of elasticity. Tracing the history of this subject in his magnum opus *History of Strength of Materials* [1], Timoshenko cites the contributions of Leonardo da Vinci, Galileo, Mariotte, Euler, Bernoulli, Young, Saint-Venant, Foppl and Prandtl.

General student unrest in Petersburg in the year 1904-1905 propelled Timoshenko to Germany. As it turned out, the torsion bending problem that he took up at the suggestion of Prandtl formed Timoshenko's dissertation at Kiev in 1907. Timoshenko narrated this exciting history in his autobiography *As I Remember* [2] (*Box* 1).

Timoshenko traces further development on this topic citing Wagner in 1929 Kappus in 1938 and Vlasov in 1940 who developed more rigorous analyses. The coupling of torsion and bending in short beams suggests a strategy for designing sensitive and compact wind tunnel balances for sensing aerodynamic forces and moments [3]. A wind tunnel balance is an intricately shaped elastic member inserted between the model and a rigid sting as shown in *Figure* 1.

Keywords
Wind tunnel, balance, torsion bending.

The aerodynamic forces acting on the model are transferred to the sting via the balance. Strain gauges attached to the balance

Box 1.

"The 1904-1905 school year was an unquiet one... . Since it was difficult to work quietly in Petersburg, and the Polytechnic Institute was closed, I decided to use the time for studying at one of the German universities. There had accidentally come to my attention a note concerning L Prandtl's work on the stability of beams. Aspects of the stability of elastic systems had interested me ever since Yasinsky's lectures at the Ways of Communication Institute, so I made up my mind to do some work under Prandtl at the University of Göttingen where at this time the ideas of F Klein had definitely triumphed. In the school of Philosophy they had established an Institute of Applied Mathematics (under Professor Runge), Institute of Applied Mechanics (under Professor Prandtl), Institute of Electrical Engineering (under Professor Simon). Klein's basic idea was to forge a closer link between mathematics and its applications. Several courses of an applied nature had been announced, also a seminar (presided over by Klein) in which all the professors of applied sciences, and senior students interested in applications of mathematics in engineering, were to take part. All these ideas appealed to me very much, and I went to Göttingen with high hopes. The first weeks at Göttingen were difficult for me. I had not allowed for the fact that the end of April and beginning of May are colder there than in Kiev, and that the houses were poorly heated. To study in overcoat and cap, hands and feet freezing, was hard. I soon discovered the existence of the *Lesezimmer*, a special library for senior students of mathematics, which contained the principle mathematics books and reference works. For two marks, a student got a key to this library, could work there in warmth, but the main thing was that, with no formalities at all, he could use all the books on the shelves. This was a very useful institution, and there I gained my first acquaintance with books close to my field about which I had previously known nothing.

At the start of classes I signed up for Abraham's lectures on practical differential equations, V Voigt's lectures on problems in mechanics, and for laboratory work on strength of materials under Prandtl. I worked also under Klein in his seminar, devoted that semester to electrical engineering. I decided not to take too many lecture hours, my main job being the work under Prandtl on my dissertation.

Prandtl suggested first experimental work, but such work did not suit me. Prandtl's laboratory was poorly equipped; one had to do everything oneself. Under such conditions the whole semester would have been spent just in preparation, and no actual work would have been done. It was more practical to do the experiments in Petersburg, where we had not only a laboratory but a mechanic and workshops. I requested theoretical work, and Prandtl proposed that I continue his own dissertation. He had been investigating the lateral buckling in flexure of a beam of narrow rectangular section, though for practical purposes, of course, it was more important to study the lateral stability of an I-beam. In that case one had to start with the torsion of an I-beam. Here for the first time it was found that, for solution of this problem, the Saint-Venant principle is not applicable. Obviously, the angle of torsion depends not only on the torsional moment and torsional rigidity of the beam but on the manner in which its ends are fastened. If the beam end is rigidly fixed, in torsion the beam's flanges will obviously bend, and this bending has to be allowed for. It took me about two weeks to figure out how to allow for this bending, to realize that the torque is

Box 1 continued ...

Box 1 continued ...

counterbalanced by the same stresses as in ordinary torsion, added to the moment produced by the shear forces resulting from the buckling of the I-beam's flanges. Once this was understood, writing an equation for the torsion was no longer difficult.

Instead of the usual torsion equation $M^t = C\phi'$, I got the equation $M^t = C\phi' - D\phi'''$, wherein D depends on the bending rigidity of the flanges and on the height of the beam section. With this equation it ceased to be difficult to investigate special cases, and to show that the Saint-Venant principle applies only in the case of very long beams.

Having solved the torsion problem, I could now tackle the question of a beam's stability. The work plan was the same as in Prandtl's dissertation, with one complicating exception: Instead of the second-order equation arrived at by Prandtl, I had to deal with a fourth-order equation, achieving its integration with the aid of series. This required much arithmetic labor, but a result was obtained, and I still remember my joy when I finally got it. It turned out that, as the beam's length increased, my result came closer and closer to Prandtl's, as one would expect. Now all I had to do was calculate the critical load for the various special cases. By the end of the semester I already had a whole series of results that I could show to Prandtl. At that time Prandtl, no longer interested in structural mechanics, was busy with the theory of high-velocity gas flows. He approved my results, and later, after many years, told me that I had been a good student, because I had not pestered him with questions and had worked independently.

My main purpose in Göttingen was achieved, and I completed most of the work which, two years later, I was to use in Kiev for my dissertation. But I got more than that from my stay in Göttingen. In Russia I had grown used to professor's repeating the same lectures year after year. Here I saw how a science expands, how new branches of it evolve. For instance, my comrade at the Petersburg Polytechnic, Nikolai (Yevgeniy Leopol's dovich), in Göttingen took a course with D Hilbert on integral equations. This was the very first time that such a course had ever been given. At the same time, at the Physics Institute, there was a seminar on electron theory – again, an entirely new field."

measure the forces and moments. It is important to highlight the demanding and often conflicting requirements that need to be simultaneously met by a wind tunnel balance. It has to be compact for easy accommodation within the models being tested. It should be sensitive exhibiting a linear response to the applied forces and moments. At the same time a wind tunnel balance should be stiff and strong to endure shocks and ensure long life.

While coping with all these conflicting demands, a designer finds that sensing axial forces and rolling moments pose the greatest challenge. Restricting our discussion to the measure-

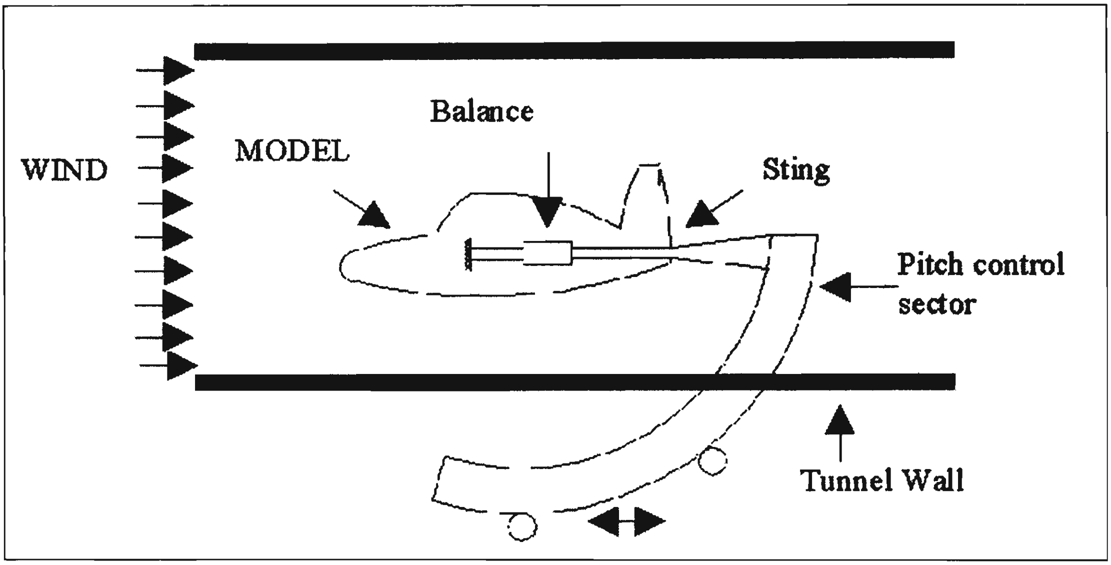

Figure 1. Wind tunnel arrangement.

ment of rolling moments, we analyse the response of a short beam of rectangular cross-section to a torque. If the two ends of the beam are left free the situation corresponds to pure torsion studied by St. Venant. However, the free ends warp and no longer remain planar and parallel to each other. Only for the special case of a circular section that warping is absent and the two ends remain plane and parallel.

Returning to the rectangle, suppose the ends are not left free and warping is prevented. This leads to considerable axial strain in the beam near the fixed ends which diminishes exponentially moving away from the ends. In the case of a short beam, however, this axial strain exerts a dominating influence throughout offering a unique means for torque measurement. The axial rate of twist ($\alpha = \theta = \mathrm{d}\theta/\mathrm{d}z$), which is uniform in St. Venant's torsion is rendered non-uniform. The rate of twist is zero at the fixed ends and peaks at midsection. Further details of this analysis are given in [3]. Briefly, re-writing Timoshenko's expression for the applied torque in the form attributed to Wagner and Kappus.

$$T = GJ\,\theta' - E\,\Gamma\theta''' \qquad (1)$$

Timoshenko
Zurich 1926

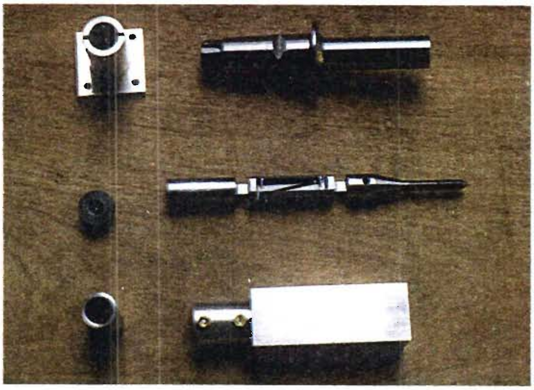

Figure 2. Wind tunnel balance.

In the above equation, E is Young's modulus, G is shear modulus, J and Γ are torsion-bending constants determined by the sides of the rectangle. Setting $\lambda^2 = GJ / EG$, (1) can be recast as

$$\alpha - \lambda^2 \alpha = T/EG. \qquad (2)$$

For a short beam of length $2l$, the correct solution with vanishing rate of twist at the fixed ends is

$$\alpha = \frac{T}{GJ}\left[1 - \frac{\cos h(\lambda\, z)}{\cos h(\lambda\, l)}\right]. \qquad (3)$$

More details of the analysis and numerical results are presented in [3] from the standpoint of wind tunnel balance design to measure rolling moments. These results demonstrate the viability of using short beams for designing wind tunnel balances (*Figure* 2). In conclusion, Timoshenko's early work with Prandtl paved the way for Wagner and Kappus theory of torsion bending.

A simple interpretation for this theory can be developed by experimenting with spring or a telephone cord. The cord shortens for a given direction of twist, and unwinds and elongates in the other direction. This winding and unwinding alters the pitch and the coil diameter. Suppose the cord or the spring is required to maintain a constant length during the application of the torque, an axial force is also necessary. This effect was considered by Thomas Young way back in 1802 while conducting his famous tests to measure the Young's modulus and shear modulus. Young employed circular rods and thin wires for his experiments. A hundred years later, Prandtl initiated torsion experiments on thin walled channel sections like I-beams, and inspired Timoshenko later.

Suggested Reading

[1] S P Timoshenko, *History of Strength of Materials*, McGraw-Hill, 1953.

[2] S P Timoshenko, *As I Remember*, Van Nostrand, 1968.

[3] R Srinivasa Murthy, T S Ramamurthy and S P Govinda Raju, Analysis of Torsion of a Short Beam, *Proc. Int. Symp. on Advances in Mechanical Engineering*, Ed. T S Mruthyunjaya, pp. 1247-1253, Narosa, 1996.

Address for Correspondence

S P Govinda Raju
Department of Aerospace Engineering
Indian Institute of Science
Bangalore 560012, India.

K R Y Simha
Department of Mechanical Engineering
Indian Institute of Science
Bangalore 560012, India.