Kaun Banega Crorepati – A Million Dollars for a Mathematician

Part 1. Of Mathematics and Mathematicians

M S Raghunathan

M S Raghunathan joined the Tata Institute of Fundamental Research, Mumbai in 1960 and is presently a Professor of Eminence there.

In this two part article I will attempt to convey the flavour of some current mathematical ideas, aiming at the non-expert. In Part 1 a little about mathematics and mathematicians is discussed.

Often there is a big gap between what the general public thinks about a profession and what the professionals themselves do. I think that this gap is particularly wide in the case of mathematics as many people do not quite know what mathematicians do. People have asked me what there was left to do in mathematics – have not all theorems been already proved! There are others who think that calculating prodigies like Sakunthala Devi are mathematicians, which they are not.

'Mathematics is the queen of all sciences' – those are the oft quoted words of Carl Friedrich Gauss the greatest mathematician of all time.

This is essentially the substance of a talk given on Science Day (28 February), 2002 under the auspices of the TIFR Alumni Association.

Much the same sentiment was expressed already in the ancient Sanskrit verse (composed, one suspects, by a mathematician)

यथा शिखा मयूराणां नागानां मणयो यथा ।
तथा वेदाङ्गशास्त्राणां गणितं मूर्धनि स्थितम् ॥

which translates as

'Like the crest of the peacock and the jewel of the serpent,
Mathematics stands at the head of all sciences'.

Keywords
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Mathematics these days has a presence in a wide range of human activities. Advanced mathematics is used in fields as wide apart as astrophysics and stock-trading. Some of this interaction, especially with physics has provided inspiration that helped develop mathematics; the outstanding example of this is the invention of the calculus as a result of efforts to understand the motion of planets. However a very large part of mathematics developed like art out of a search for the beautiful; in fact great mathematicians tend to value mathematics that evolved from such a motive more than the developments resulting from interaction with other fields.

J C F Gauss
1777–1855

Gauss when he called mathematics the queen of sciences, also went on to say that number theory is the queen of mathematics; and number theory is one area which draws nothing from outside mathematics. Fourier a great French mathematician of the 19th century chided his greater contemporary Jacobi for wasting time on 'useless' mathematics. Jacobi responded with 'A savant like Fourier ought to know that the end of all science is the glory of the human intellect and under that title a question about numbers is worth as much as a question about the system of the world'.

Joseph Fourier
1768–1830

I spoke of the search for the beautiful. Words like 'beauty' are familiar words when one talks about art. But the word is also used by scientists and perhaps more often by mathematicians than other scientists. If you were thrilled by Euclidean geometry in your school days, you understand what beauty means in mathematics. Euclid's geometry does indeed connect with the real world, but the connection is tenuous. It was fascination with the beauty of geometric structures and subtle reasoning rather than any practical concerns that was the driving force for the ancient Greeks. The same drive made them pioneers in number theory as well. The first important question in number theory was raised and answered by Euclid:

Carl Jacobi
1804–1851

Question: Are there infinitely many primes?
Answer: Yes.

Let me recall that a whole number p is a prime if it is different from 1 and the only whole numbers that divide it exactly are 1 and itself. The first few primes are:

G F B Riemann
1826–1866

2, 3, 5, 7, 11, 13, 17, 19, 23, ...;

and Euclid's theorem is that this list goes on indefinitely. Most – if not all – mathematicians, consider this theorem and its proof a lovely piece of mathematics. Prime numbers fascinated Euclid and they have continued to interest generations of mathematicians over the last two millennia; and this fascination is mostly the result of mere curiosity about numbers. Euclid's theorem is elementary, but the spirit behind questions that number theorists or other pure mathematicians pose to themselves is no different. Fermat's last theorem whose proof made front page news some nine years ago was a question that haunted mathematicians for three centuries. It has no practical use whatever, at least not yet. The amazing thing is that mathematics inspired by purely aesthetic considerations has often proved to be the right tool for understanding the physical world around us, a fact pithily described by the renowned physicist Wigner as the 'unreasonable effectiveness of mathematics in the natural sciences'. A major example of this is the invention of Riemannian geometry by Bernhard Riemann (of whom I will be saying more later). The glory of the human intellect was indeed his motivation. Half a century after Riemann, his invention was to become the foundation for Einstein's relativity.

I hope I have been able to convey to some extent what the mathematicians think of their profession and themselves.

Albert Einstein
1879–1955

It is perhaps not surprising that for the outsider, mathematics should appear an esoteric pursuit; and the mathematician, a clumsy absent-minded person. Words like nerd, geek, etc. that have emanated from American college campuses (and eagerly picked up by their counterparts here) seem to have been inspired by the mathematics enthusiast. Fiction by and large seems to promote this image of the mathematician – rather like that of Professor Calculus of the *Tintin* comics. Perhaps this is just as well – better Professor Calculus than the sinister Professor Moriarty of Sherlock Holmes' stories, the well-known fictional mathematician.

There are, I suppose some mathematicians who would fit in the Professor Calculus mould, but most will not provide you the kind of unintended entertainment that he does. The one characteristic which is incomprehensible to the outsider that all mathematicians share is excitement over abstract concepts and their interactions and delight in abstruse reasoning. I suppose that is reason enough for our pursuit to be considered esoteric. It is inevitable then that most mathematicians are academics – teachers and researchers working in academic institutions and so are middle class people on the economic scale. Some, especially in western countries work in areas that can bring in substantial material rewards – Wall street routinely employs mathematics PhDs these days. But can the votary of pure mathematics entertain hopes of joining the ranks of crorepatis, especially now that KBC has come and gone without any mathematician receiving a call from Mr. Bachchan? Yes, indeed; and that without straying from – in fact only by intensifying – his esoteric pursuit. In May 2000, the Clay Mathematics Institute set up by a philanthropic US foundation came up with an offer of a prize of one million dollars each for the solution of seven mathematical problems; and a million dollars can make you a crorepati five times over. Four of the seven problems are inspired by the urge to understand purely mathematical structures; and two of those four are in number theory; the other three have their sources in fluid flow, particle physics and computer science, but are none the less exciting to the pure mathematician.

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So the new millennium brought the prospect of becoming a crorepati to the mathematician. There is a debate, at times amusingly passionate, as to which year – 2000 or 2001 – ushered in the new millennium. But I would not be wrong even if it is 2001 that did it, for in November of that year the Norwegian government announced the institution of a prize for mathematics along the lines of the Nobel Prize.

Some of you may be aware that there is no Nobel prize for mathematics. There is an interesting story circulating among mathematicians as to why. Nobel's wife ran away with a distin-

Alfred Nobel
1833–1896

guished Swedish mathematician, Mittag-Leffler; and Mittag-Leffler would have been a serious candidate for the Nobel Prize if there was one for mathematics. That prospect naturally did not go down well with Nobel and he decided against instituting a prize for mathematics. But the story is not true – for one thing, Nobel never married! But you can draw one conclusion: we mathematicians are normal in one respect; we have our share of lively malice.

The Norwegian prize is named after Henrik Abel, a towering figure in mathematics and undoubtedly the greatest from Scandinavia: 2002 is the bicentennary year of his birth. Mathematicians can now feel one-up on other scientists even if at $550,000 the monetary value of the Abel price is only half that of Nobel: For Nobel is no Abel in intellectual prowess and the prize money is not tainted by TNT!

It is ironic though that the Abel Prize will make a mathematician prosperous while Abel himself lived and died in acute poverty. He died at 27, an age when most present day mathematicians are still at their first dissertations; by then he had made several immortal discoveries. Sadly all recognition came posthumously: in comparison Srinivasa Ramanujan can be deemed to have been lucky.

Neils Henrik Abel
1802–1829

There are a few other prizes for Mathematics, but barring two, the Crawfoord Prize and the King Faisal Prize (given once in four years) their monetary value is nowhere near that of the Nobel and in any case none have the same high profile that the Nobel has. Of all these prizes the one that the mathematical community considers most prestigious is the Fields Medal which carries a cash prize of a mere 10,000 Swiss Francs (approximately $5000). Fields, a Canadian mathematician left a small legacy out of which up to four mathematicians are awarded the medal named after him once every four years at the International Congress of Mathematicians, the largest gathering of mathematicians. Incidentally a Congress is to be held this August in Beijing.

Fields Medal (Front)

Fields Medal (back)

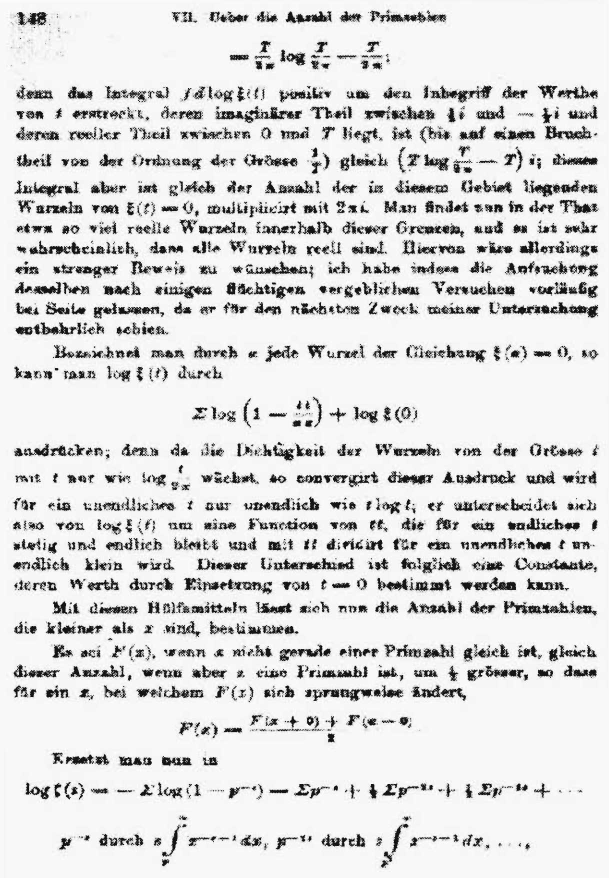
J C Fields
1863–1932

The Fields Medal is awarded only to mathematicians under the age of forty at the time of the congress. Harish-Chandra the greatest Indian mathematician since Srinivasa Ramanujan came close to getting the medal; many people think that he should have got the medal in 1958. Harish-Chandra's name is not widely known in the country – and that is reflective of the low profile of mathematics: lesser lights shine more brightly in our media. However, I should add that all of Harish-Chandra's work was carried out in the US, much of it as a professor at the famed Institute for Advanced Study in Princeton, the institution where Albert Einstein worked.

Let me get back to the Clay Prizes. I mentioned that there are seven problems. Of these 5 questions were raised by eminent mathematicians, who have in fact indicated what they themselves expect the answer to be – they have posed what mathematicians call conjectures. Leaders in the field, because of their insights are able to ask fascinating questions answering which is a challenge other experts like to take on.

Harish-Chandra
1923–1983

All the Clay problems have been around for quite some time. Among them the one that has remained unsolved for the longest period is the 'Riemann hypothesis'. Bernhard Riemann, a German mathematician is one of the all-time great figures in mathematics. He is perhaps the 19th century mathematician who had the greatest influence on the way mathematics has evolved since.



148 VII. Ueber die Anzahl der Primzahlen

$$= \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi};$$

denn das Integral $\int d \log \xi(t)$ positiv um den Inbegriff der Werthe von t erstreckt, deren imaginärer Theil zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ und deren reeller Theil zwischen 0 und T liegt, ist (bis auf einen Bruchtheil von der Ordnung der Grösse $\frac{1}{T}$) gleich $\left(T \log \frac{T}{2\pi} - T\right) i$; dieses Integral aber ist gleich der Anzahl der in diesem Gebiet liegenden Wurzeln von $\xi(t) = 0$, multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.

Bezeichnet man durch α jede Wurzel der Gleichung $\xi(\alpha) = 0$, so kann man $\log \xi(t)$ durch

$$\Sigma \log \left(1 - \frac{tt}{\alpha\alpha}\right) + \log \xi(0)$$

ausdrücken; denn da die Dichtigkeit der Wurzeln von der Grösse t mit t nur wie $\log \frac{t}{2\pi}$ wächst, so convergirt dieser Ausdruck und wird für ein unendliches t nur unendlich wie $t \log t$; er unterscheidet sich also von $\log \xi(t)$ um eine Function von tt, die für ein endliches t stetig und endlich bleibt und mit tt dividirt für ein unendliches t unendlich klein wird. Dieser Unterschied ist folglich eine Constante, deren Werth durch Einsetzung von $t = 0$ bestimmt werden kann.

Mit diesen Hülfsmitteln lässt sich nun die Anzahl der Primzahlen, die kleiner als x sind, bestimmen.

Es sei $F(x)$, wenn x nicht gerade einer Primzahl gleich ist, gleich dieser Anzahl, wenn aber x eine Primzahl ist, um $\frac{1}{2}$ grösser, so dass für ein x, bei welchem $F(x)$ sich sprungweise ändert,

$$F(x) = \frac{F(x+0) + F(x-0)}{2}.$$

Ersetzt man nun in

$$\log \zeta(s) = -\Sigma \log(1 - p^{-s}) = \Sigma p^{-s} + \frac{1}{2}\Sigma p^{-2s} + \frac{1}{3}\Sigma p^{-3s} + \cdots$$

$$p^{-s} \text{ durch } s\int_p^\infty x^{-s-1}dx,\ p^{-2s} \text{ durch } s\int_{p^2}^\infty x^{-s-1}dx,\ \ldots,$$

A page from Riemann's paper

I mentioned that Euclid had shown that there are infinitely many prime numbers. This led mathematicians to ask how they are distributed among all whole numbers. In particular what proportion of whole numbers less than a given number x are primes? In 1859 Riemann published a short (9 page) paper in the *Journal Monatsberichte der Berliner Akademie* where he related this question to the behaviour of a function now called the Riemann Zeta function of a complex variable. He made a conjecture regarding the set of points where this function takes the value 0 and this is known as the Riemann hypothesis. The truth of the hypothesis has many striking consequences in number theory.

The problem has fascinated many of the leading minds in mathematics ever since Riemann formulated it, among them the British mathematician Hardy whose name is familiar to us because of his association with Ramanujan.

Once on the eve of taking a boat to cross the choppy English channel from Europe to England Hardy dashed off a postcard to his friend Herald Bohr telling him that he had proved the Riemann hypothesis!

> *"You must know that Hardy had a running feud with God. In Hardy's view God had nothing more important to do than frustrate Hardy. This led to a sort of insurance policy for Hardy one time when he was trying to get back to Cambridge after a visit to [Herald] Bohr in Denmark. The weather was bad and there was only a small boat available. Hardy thought there was a real possibility the boat would sink. So he sent a postcard to Bohr saying, "I proved the Riemann Hypothesis. G.H. Hardy." That way if the boat sank, everyone would think that Hardy had proved the Riemann Hypothesis. God could not allow so much glory for Hardy so he could not allow the boat to sink."*
>
> (George Polya, quoted in *Out of the Mouths of Mathematicians*, by R Schmalz)

Srinivasa Ramanujan
1887–1920

G H Hardy
1877–1947

David Hilbert
1862–1943

David Hilbert, twentieth century's greatest mathematician, in a famous address to the International Congress in 1900, listed the Riemann hypothesis as one of the most important unsolved problems; a whole century has not changed that status.

Chronologically, the Clay problem that appeared next on the mathematical scene is the Poincaré conjecture. Henri Poincaré who lived 12 years into the twentieth century is arguably the greatest figure produced by France in the last century and a half. As a mathematician he is in the same league as Gauss and Riemann. He was a versatile mathematician who has left his indelible imprint on virtually every branch of mathematics.

Jules Henri Poincare
1854–1912

His work constitutes a treasure house from which mathematicians pick out gems every now and then try to polish them further. Topology, one of the central branches of contemporary mathematics was essentially his creation. The Poincaré conjecture, one of the Clay problems is a problem in topology. I will talk more about it shortly.

The motion of fluids is governed by certain differential equations known as Navier–Stokes equations. One of the Clay problems seeks the proof of any one of four assertions concerning solutions of this equation, one of the assertions being a conjecture of a French mathematician Jean Leray. Leray, one of the most original mathematicians of the twentieth century was

Jean Leray
1906–1998

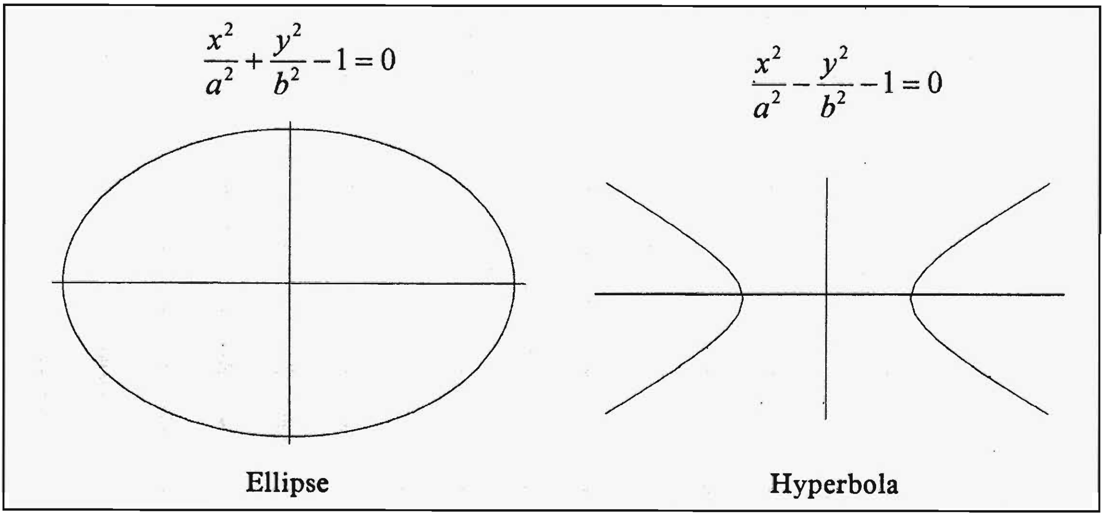
Algebraic geometry originates from the study of sets of points where a given bunch of polynomial functions on a n-dimensional Euclidean space take the value 0 such as ellipses and hyperbolas.

responsible for much of the progress in the study of the Navier–Stokes equation in the first half of the 20th century, a 1934 paper of his being a landmark.

Leray's own work was all theoretical but fluid mechanics has military applications. During the Nazi occupation of France Leray feared that if he were identified as an expert in the area, he might be coerced into helping the German war machine and so he declared himself a topologist: Hitler's generals had no use for topology. He then went on to produce some fantastically original work in topology demolishing long standing problems and introducing breathtaking new techniques. After the war he went back to differential equations.

Algebraic geometry is an important area of mathematics at which a very large work-force is pegging away. It is incidentally an area in which TIFR has made a big name. The subject originates from the study of sets of points where a given bunch of polynomial functions on a n-dimensional Euclidean space take the value 0 such as ellipses and hyperbolas.

One of the Clay Prizes is for settling a problem in this area known as Hodge's conjecture. Sir William Hodge was a Cambridge mathematician and one of those on whom rested Britain's




William V D Hodge 1903-1975

Alexander Grothendieck

claims to being a leading mathematical power in the twentieth century.

Once again I will not be able to say anything as to what the problem is about because of its highly technical nature. Inspired by the Hodge conjecture, Alexander Grothendieck who dominated algebraic geometry in the second half of the last century has drawn up a programme of investigation that has set the agenda for all the leaders in the field intensifying further the interest in the conjecture. TIFR had the privilege of hosting Grothendieck at an International Colloquium organised here in Mumbai in 1968 where he presented some of these ideas.

B J Birch

Peter Swinnerton-Dyer

The fifth Clay problem is again a conjecture, made in 1965, jointly by two eminent British mathematicians, both living – B J Birch of Oxford and Sir H P F Swinnerton-Dyer of Cambridge. The problem belongs to number theory and is a statement about what are known as Elliptic curves a subject which has its origins in the work of, among others, Abel. Elliptic curves also had an important role in the proof of Fermat's last theorem. Any attempt at describing the conjecture itself to the non mathematician is hopeless: a great deal of sophisticated background is needed for the formulation. Birch visited TIFR in 1968 to give an invited talk at the same meeting where Grothendieck spoke on matters related to the Hodge conjecture. Swinnerton-Dyer is a chess as well as a bridge player with an international standing.

He has held important administrative positions – among them Vice Chancellorship of Cambridge University and Chairmanship of the University Grants commission of UK.

The two remaining problems are not associated with any names in particular. One of them draws its inspiration from computer science and the other from particle physics. I do not know much about these problems, not even the kind of trivia I could mention in connection with the others and so will say nothing further about them.

In the next part of this article I will try to explain one of the above problems – the Poincaré conjecture.

Address for Correspondence
M S Raghunathan
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400 005, India.

Please Note

Modified *Figure* 2.

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