Think It Over

This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

A Problem in Number Theory

M N Deshpande
π, 22 Panchdeep Nagar Wardha Road,
Nagpur 440 025
Email:
dpratap@nagpur.dot.net.in

Find a way of generating infinitely many triples (a, b, c) of positive integers, with the condition that $ab+1$, $bc+1$ and $ca+1$ are perfect squares, and $(a+b)/c = 2/3$.

Solution First we show how to generate infinitely many triples (a, b, c) of positive integers such that $ab+1$, $bc+1$ and $ca+1$ are all squares. Let a, b, c be chosen so that $ab+1$ is a square, say x^2; let $c = a+b+2x$. Then

$$bc+1 = b(a+b+2x)+1 = b^2+2bx+(ab+1) = b^2+2bx+x^2 = (b+x)^2,$$

and we find in the same way that $ac+1$ is a square.

Thus, starting with $(a, b) = (1, 8)$ we get $c = 15$; starting with $(a, b) = (1, 15)$ we get $c = 24$; and so on. We see in this way how infinitely many triples (a, b, c) of positive integers may be generated, having the property that $ab+1$, $bc+1$ and $ca+1$ are all squares.

Now we impose the extra condition $(a+b)/c = 2/3$, or $3(a+b) = 2c$. Obviously $a+b$ must now be even. We fix x so that

$$\frac{a+b}{a+b+2x} = \frac{2}{3}, \quad \text{giving } a+b = 4x \text{ and } c = \frac{3(a+b)}{2}.$$

We now get

$(a+b)^2 = 16x^2 = 16(ab+1)$,

therefore $a^2 + b^2 - 14ab - 16 = 0$.

The equation may be rewritten as

$$a^2 - 14ab + 49b^2 = 16 + 48b^2,$$

or $(a-7b)^2 = 16(1+3b^2)$. So we must find integer values of b for which $1 + 3b^2$ is a square, say d^2

The equation $1 + 3b^2 = d^2$ is an example of 'Pell's equation'. Such equations (or more generally, the equation $1 + Nx^2 = y^2$ where N is a given positive integer) were studied long back by the Indian mathematicians Āryabhatta and Bhāskarāchāryā, Later they were studied by Fermat, and still later by Euler and Lagrange, and somewhere along the way the equation got wrongly named as 'Pell's equation'. In the *Bījaganitam*, Bhāskarāchāryā gives a complete algorithmic solution to the problem.

The integer solutions of $1 + 3b^2 = d^2$ are listed below:

b	1	4	15	56	209	780
d	2	7	26	97	362	1351

The underlying recursion relation is clearly visible: if (b_n, d_n) denotes the n^{th} pair, then $(b_1, d_1) = (1, 2)$, $(b_2, d_2) = (4, 7)$, and

$$(b_{n+2}, d_{n+2}) = 4 \quad (b_{n+1}, d_{n+1}) - (b_n, d_n)$$

From (b, d) we get (a, b, c) via $a = 7b+4d$, $c = a+b+4x = 3(a+b)/2$. The corresponding (a, b, c) list is displayed below:

a	15	56	209	780	2911	10864
b	1	4	15	56	209	780
c	24	90	336	1254	4680	17466

The underlying recursion is the same as given earlier:

$$(a_{n+2}, b_{b+2}, c_{n+2}) = 4 \quad (a_{n+1}, b_{n+1}, c_{n+1}) - (a_n, b_n, c_n),$$

with $(a_1, b_1, c_1) = (15, 1, 24)$ and $(a_2, b_2, c_2) = (56, 4, 90)$.

We have accomplished the task we set out to do. It is interesting to ask whether there are any solutions other than the ones given above.

A Geometric Dissection Problem

Find a way of cutting up a regular octagon into twelve congruent isosceles triangles and one square.

Solution

We shall show, more generally, how to cut up a regular $2n$-gon $\mathcal{P}$ into $n(n-1)$ congruent isosceles triangles and a regular polygonal figure $\mathcal{Q}$ with $n(n-3)$ sides; here $n \geq 3$. For $n > 4$ the inner polygon $\mathcal{Q}$ is non-convex. The given problem corresponds to the case $n = 4$; the inner polygon is a square in this case.

Let the vertices of the polygon be labelled in sequence 1, 2, 3, , $2n$; let s be its side. Construct $n(n-1)$ congruent copies of an isosceles triangle whose equal sides have length s and whose apex angle is $\theta = \pi/n$ radians.

Each interior angle of the polygon is $(n-1)\theta$, so $(n-1)$ of these triangles can be placed inside $\mathcal{P}$ without overlap with their apexes at vertex 1. This can be repeated for vertices 3, 5, .., $2n-1$ (alternate vertices only). In this way all the $n(n-1)$ triangles get 'used up', and there is no overlap; the central region left uncovered is a regular polygonal figure $\mathcal{Q}$ having $n(n-3)$ sides.

Sketches for the cases $n = 4$ and $n = 5$ are displayed in *Figures* 1 and 2 respectively.

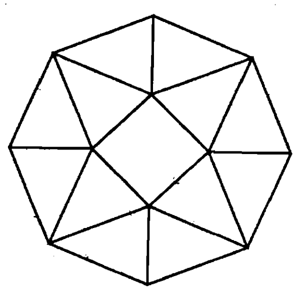

Figure 1. n=4.

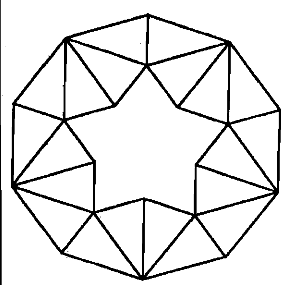

Figure 2. n=4.