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Symmetry on a Sphere

Area of a Spherical Cap

Consider a circle drawn in the usual manner using a compass, with the radius set at L. The area of the circle is, of course, πL^2. Now, keeping the gap between the legs intact, place the point of the compass on a sphere and trace out a circle on the sphere (see *Figure* 1). What will be the area of the spherical cap thus formed? Surprisingly, the same formula applies! – the area is πL^2, as earlier.

The proof is not too difficult. In fact there are several proofs available. Perhaps the simplest way is to use the method used by Archimedes to show that the surface area of a sphere of radius R is $4\pi R^2$. We shall however use integration.

Keywords
Spherical cap, spherical triangle, polar angle.

Let a spherical polar coordinate system be used; let θ be the polar angle (the complement of the latitude), and let ϕ be the longitudinal angle. The element of area on the surface of the sphere is then $(R\,d\theta)\,(R\sin\theta\,d\phi) = R^2\sin\theta\,d\theta\,d\phi$. For the spherical cap under discussion, we have $0 \leq \phi \leq 2\pi$ and $0 \leq \theta \leq \theta_0$ where

$$\sin\frac{\theta_0}{2} = \frac{L/2}{R},$$

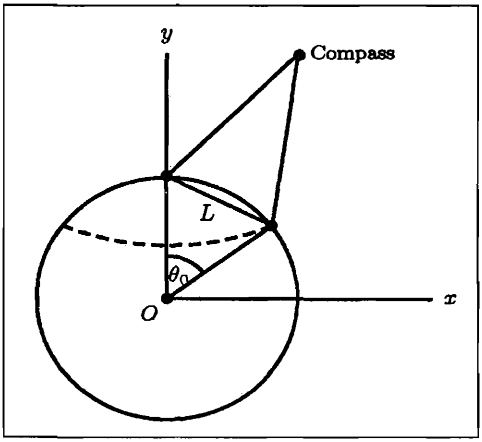

Figure 1. Area of a spherical cap.

thus

$$\theta_0 = 2\sin^{-1}\frac{L}{2R}.$$

It follows that the required area is equal to

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\sin^{-1}L/2R} R^2 \sin\theta \, d\theta \, d\phi$$

$$= 2\pi R^2 \left[1 - \cos\left(2\sin^{-1}\frac{L}{2R}\right)\right]$$

$$= 4\pi R^2 \sin^2 \sin^{-1}\frac{L}{2R} = \pi L^2.$$

So the area of the cap is independent of the radius of the sphere! This is clearly a special property of the sphere.

1. The area of the spherical cap is independent of the radius of the sphere, while the circumference of the edge of the cap depends on the radius.

2. Area of the circle on the plane with radius L and the area of the spherical cap with compass length L are the same, while the circumference of the circle and the circumference of the edge of the spherical cap are different.

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Area of a Spherical Triangle

The idea of a triangle on a spherical surface (its sides are arcs of 'great circles') sounds forbidding, and one may anticipate that the formula for the area of such a triangle is given by some complicated expression. Instead, the formula turns out to be simple and elegant – simpler than the corresponding formula for a plane triangle! Indeed, we have the following: *The area of a spherical triangle on a sphere of radius R is equal to*

$$(\text{sum of the angles of the triangle} - \pi)\, R^2$$

As earlier, a variety of proofs are available, and the reader may wish to find a non-calculus proof, using only simple geometrical ideas. We shall once again give a

The area of a spherical triangle on a sphere of radius R is equal to (sum of the angles of the triangle – π) R^2.

proof that uses integration. Recall firstly that great circles are formed by the intersection of the surface of the sphere with planes passing through the center of the sphere.

We take the sphere to have radius R and to be centered at the origin $O(0,0,0)$, one vertex of the the triangle to be the 'north pole', $A\,(0,0,R)$ (see *Figure* 2), and the two planes passing through the north pole to be

$$y = 0, \qquad -x\sin\alpha + y\cos\alpha = 0.$$

Let the third side of the triangle correspond to the plane

$$z = ax + by,$$

where a, b are constants. The normal directions of the three planes are, in vector form,

$$(0,1,0), \qquad (-\sin\alpha, \cos\alpha, 0), \qquad (a, b, -1).$$

The lengths of these three vectors are, respectively,

$$1, \qquad 1, \qquad \sqrt{a^2+b^2+1},$$

and their pairwise scalar products are

$$\cos\alpha, \qquad b, \qquad -a\sin\alpha + b\cos\alpha;$$

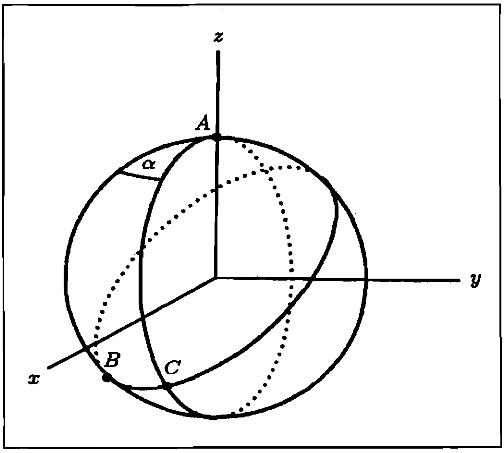

Figure 2. Area of a spherical triangle.

so the angles of the spherical triangle are α, β and γ, where β and γ are given by

$$\cos\beta = \frac{b}{\sqrt{1+a^2+b^2}}, \qquad \cos\gamma = \frac{a\sin\alpha - b\cos\alpha}{\sqrt{1+a^2+b^2}}.$$

We now transform to spherical coordinates; with θ and ϕ having the same meanings as earlier, the conversion is achieved via

$$z = R\cos\theta, \qquad x = R\sin\theta\cos\phi, \qquad y = R\sin\theta\sin\phi.$$

The relation $z = ax + by$ yields $\cot\theta = a\cos\phi + b\sin\phi$, and it follows that the required area is]

$$\begin{aligned}
&\int_{\phi=0}^{\alpha}\int_{\theta=0}^{\cot^{-1}(a\cos\phi+b\sin\phi)} R^2\sin\theta\, d\theta\, d\phi \\
= \;& R^2\int_{\phi=0}^{\alpha}\left[1-\cos\cot^{-1}(a\cos\phi+b\sin\phi)\right] d\phi \\
= \;& \alpha R^2 - \int_{\phi=0}^{\alpha}\left[\frac{(a\cos\phi+b\sin\phi)}{\sqrt{1+(a\cos\phi+b\sin\phi)^2}}\right] R^2\, d\phi \\
= \;& \alpha R^2 - \int_{\phi=0}^{\alpha}\frac{\sqrt{a^2+b^2}(\cos\delta\cos\phi+\sin\delta\sin\phi)R^2}{\sqrt{1+(a^2+b^2)(\cos\delta\cos\phi+\sin\delta\sin\phi)^2}}\, d\phi,
\end{aligned}$$

where for convenience we have introduced an auxiliary angle δ defined by

$$\cos\delta = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\delta = \frac{b}{\sqrt{a^2+b^2}}.$$

The required area may now be written as

$$\begin{aligned}
&\alpha R^2 - \int_{\phi=0}^{\alpha}\frac{R^2\cos(\phi-\delta)\, d\phi}{\sqrt{\frac{1+a^2+b^2}{a^2+b^2}-\sin^2(\phi-\delta)}} \\
&= \alpha R^2 - \left[\sin^{-1}\left(\frac{\sqrt{a^2+b^2}\sin(\alpha-\delta)}{\sqrt{1+a^2+b^2}}\right) + \right. \\
&\qquad \left.\sin^{-1}\left(\frac{\sqrt{a^2+b^2}\sin\delta}{\sqrt{1+a^2+b^2}}\right)\right] R^2
\end{aligned}$$

If the sum of the angles of a spherical triangle is held constant, then its area stays constant. (This clearly does not hold in the case of plane triangles.

$$= \left[\alpha - \pi + \cos^{-1}\left(\frac{\sqrt{a^2+b^2}\sin(\alpha-\delta)}{\sqrt{1+a^2+b^2}}\right) + \cos^{-1}\left(\frac{\sqrt{a^2+b^2}\sin\delta}{\sqrt{1+a^2+b^2}}\right)\right] R^2$$

$$= \left[\alpha + \cos^{-1}\left(\frac{a\sin\alpha - b\cos\alpha}{\sqrt{1+a^2+b^2}}\right) + \cos^{-1}\left(\frac{b}{\sqrt{1+a^2+b^2}}\right) - \pi\right] R^2$$

$$= (\alpha+\beta+\gamma-\pi)R^2.$$

This proves the claim.

Corollaries

Several interesting corollaries drop out rather easily from the above result. We list some of these below.

1. *If the sum of the angles of a spherical triangle is held constant, then its area stays constant.* (This clearly does not hold in the case of plane triangles.)

2. *The sum of the angles of a spherical triangle is never less than π and never greater than 5π.* (This is so because the area of the triangle is non-negative, and does not exceed the surface area of the whole sphere.)

3. *The area of a n-sided spherical polygon drawn on a sphere of radius R is given by*

$$[\text{sum of the angles of the polygon} - (n-2)\pi]\, R^2.$$

(This follows from the fact that the n-sided polygon can be partitioned into $(n-2)$ spherical triangles.)

4. *If the sum of the angles of a n-sided spherical polygon is held constant, then its area stays constant.*

5. *The sum of the angles of a n-sided spherical polygon is never less than $(n-2)\pi$ and never greater than $(n+2)\pi$.*