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Minimum Time of Travel

Introduction

The following type of problem is discussed in many textbooks of dynamics. Suppose, a train starts from rest with a uniform acceleration, attains a certain speed and thereafter retards with a uniform retardation finally to stop at the next station. In most of the problems the train is considered to have moved with that speed for some time before taking up the retardation. How does one determine the minimum time of travel? Given the applied force per unit mass, the resistive force per unit mass due to braking, the distance between the stations and also the friction between the rails and the wheels, our aim here is to determine the minimum time of journey by the train.

Let us first formulate the problem dynamically.

It is obvious that the train starts from rest at station, say, A applying a pull f_1 per unit mass. After some time it attains a speed. Let it move with this speed for some time and thereafter it applies a brake with resistive force f_2 per unit mass with a view to stop at the next station B, where as the frictional force between the rails and the wheels is f per unit mass and the motion of the train is rectilinear. In this article is aimed at determining how long the train should accelerate, how long it should move with uniform speed and how long it should retard before coming to rest at the next station so as to yield a minimum time of journey between the two stations, say, X distance apart.

Solution to the Problem

Keywords
Minimum time of travel, resistive force.

Let the train attain a speed v in time t_1 after travelling a distance x_1. Since it moves with resulting acceleration $(f_1 - f)$, starting from rest at station A, considering its rectilinear motion we get

$$v = (f_1 - f)t_1, v^2 = 2(f_1 - f)x_1 \tag{1}$$

Let the train travel a distance x with uniform velocity v for time t before it starts retarding so as to halt at the next station. Then we have

$$x = vt. \tag{2}$$

Thereafter since the train moves with resulting retardation $(f_2 + f)$ to cover a distance, say, x_2 before it comes to rest at the next station B, one gets

$$v = (f_2 + f)t_2, v^2 = 2(f_2 + f)x_2. \tag{3}$$

If T be the total time of travel by the train, (1) and (3) lead to

$$T = v\left(\frac{1}{f_1 - f} + \frac{1}{f_2 + f}\right) + t \tag{4}$$

$$X = \frac{v^2}{2}\left(\frac{1}{f_1 - f} + \frac{1}{f_2 + f}\right) + vt. \tag{5}$$

Eliminating t between (4) and (5) we get T as a function of v:

$$T = \frac{X}{v} + \frac{v}{2}\left[\frac{1}{f_1 - f} + \frac{1}{f_2 + f}.\right] \tag{6}$$

This suggests that there exists a minimum time of travel by the train between the two stations for which it has to attain a certain speed v_{opt} because $\frac{dT}{dv} = 0$ gives

$$-\frac{X}{v^2} + \frac{1}{2}\left(\frac{1}{f_1 - f} + \frac{1}{f_2 + f}\right) = 0$$

$$\text{i.e., } v_{\text{opt}} = \left(\frac{2X}{\frac{1}{f_1 - f} + \frac{1}{f_2 + f}}\right)^{1/2} \tag{7}$$

and also $\frac{d^2T}{dv^2} > 0$ and in consequence of (5),

$$t_{\text{opt}} = 0. \tag{8}$$

This reveals that to minimize this journey time the train has to accelerate to a maximum velocity given by (7) and, thereafter, has to retard to zero velocity at the next station i.e. with no travel with uniform velocity during the journey.

However, utilizing (1), (3), (6) and (7) we have

$$(x_1)_{\text{opt}} = \frac{X}{\frac{1}{f_1-f}+\frac{1}{f_2+f}}\left(\frac{1}{f_1-f}\right) \tag{9}$$

$$(t_1)_{\text{opt}} = \left(\frac{2X}{\frac{1}{f_1-f}+\frac{1}{f_2+f}}\right)^{1/2}\frac{1}{f_1-f} \tag{10}$$

$$(x_2)_{\text{opt}} = \left(\frac{X}{\frac{1}{f_1-f}+\frac{1}{f_2+f}}\right)\left(\frac{1}{f_2+f}\right) \tag{11}$$

$$(t_2)_{\text{opt}} = \left[\frac{2X}{\left(\frac{1}{f_1-f}+\frac{1}{f_2+f}\right)}\right]^{1/2}\frac{1}{f_2+f} \tag{12}$$

Hence the minimum total time of travel between two stations from rest to rest by the train is

$$\begin{aligned} T_{\min} &= (t_1)_{\text{opt}} + (t_2)_{\text{opt}} \\ &= \left[2X\left(\frac{1}{f_1-f}+\frac{1}{f_2+f}\right)\right]^{1/2} \end{aligned} \tag{13}$$

What has been established by the foregoing discussion? It is established that if a vehicle undertakes to go from one position at rest to another position at rest in a rectilinear path with a given acceleration and with a given retardation then it can complete the journey in a minimum time without moving with uniform speed for any part of the journey.

Here is a geometrical picture of the situation. Say the train starts with a fixed acceleration f_1 (including friction) and accelerates upto a velocity v, moves with constant velocity for sometime and then slows down with a fixed deceleration f_2 to zero velocity at the fixed destination. In a plot of $v(t)$ versus t, the above situation will look like.

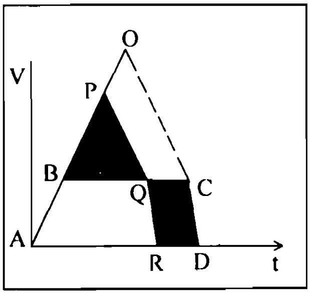

Figure 1.

AB : motion with constant acceleration f_1.

BC : motion with constant velocity v.

CD : motion with constant deceleration f_2.

The curve encloses a trapezium $ABCD$. Now the problem is to minimize the time taken, i.e., have the smallest possible AD (intercept on t axis), with the following constraints:

• the slopes of AB and CD are fixed.

• the area under the curve representing the distance between the stations is fixed.

The optimal solution is represented by the trajectories AQR (in velocity space). The time AD is reduced to AR, at the expense of adding the area of the triangle BPQ to the area under the curve in *Figure* 1, $RQCD$. To satisfy the constraint, this gain and loss in area must exactly compensate each other. The analytically derived solution has exactly this property, and the triangle AQR reflects the fact that to minimize the time, the period of travel with constant velocity must be zero, i.e., geometrically the $v(t)$ versus t graph should enclose a triangle as opposed to a trapezium.

Suggested Reading

[1] A S Ramsey, *Dynamics*, Part 1, Cambridge University Press, London, pp. 43-50, 1962.

[2] M Ray and G C Sharma, *A textbook on dynamics*, S Chand and Company Ltd., New Delhi, pp. 39-47, 1990.