colliding with a lighter ball A, both travelling to the left; see *Figure* 5. Even qualitatively, we can see that this cannot result in the lighter ball coming to rest after collision. So what is wrong?

Remember that here we want $u_1 < 0$ and $v_1 = 0$. But this violates the inequality (30), where we require $v_1 < u_1$ (which means v_1 will be more negative). Thus it is not possible to satisfy all these three inequalities together, and hence this solution is discarded.

What about symmetry between the two balls? It is a collision between two balls, and we might replace A by B and B by A. But we have broken the symmetry when we required $v_1 = 0$, that is we wanted the first ball A (the one on left) to come to rest after collision. If we want the second ball B (the one on right) to come to rest, that is $v_2 = 0$, with $m_1 \neq m_2$, then it will be possible only if they are travelling to the left, and *not* if they are travelling to the right. So the symmetry is restored again in that sense.

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How Far Apart are Primes? Bertrand's Postulate

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It is well-known that there are arbitrarily large gaps between primes. Indeed, given any natural number n, the numbers $(n+1)!+2, (n+1)!+3, \quad (n+1)!+(n+1)$ being large multiples of $2, 3, \quad , n+1$ respectively, are all composite numbers.

Let us now ask ourselves the following question. If we start with a natural number n and start going through the numbers $n+1, n+2$ etc., how far do we have to go

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Although there are arbitrarily large gaps between primes, it is nevertheless true that 'there is regularity in the distribution of primes'.

before hitting a prime? Trying out the first few numbers, we see that

$$1 \to 2, 2 \to 3, 3 \to 5, 4 \to 5, 5 \to 7, 6 \to 7, 7 \to 11 \text{ etc.}$$

Thus, it seems that we need to go 'at most twice the distance' i.e., we seem to be able to find a prime between n and $2n$ for the first few values of n. But there is absolutely no pattern here. In fact, although we have seen above that *there are arbitrarily large gaps between primes*, it is nevertheless true that *'there is regularity in the distribution of primes'* It is this fascinating clash of tendency which seems to make primes at once interesting and intriguing. It turns out indeed to be true that: *there is always a prime between n and 2n. This statement, known as Bertrand's postulate*, was stated by Bertrand in 1843 and proved later by Chebychev in 1852. Actually, Chebychev proves a much stronger statement which was further generalised to yield a fundamental fact about the prime numbers known as the prime number theorem.

Proving the prime number theorem is beyond the scope of this article but stating it certainly lies within it. For a positive real number x, let us denote by $\pi(x)$, the number of primes which do not exceed x. The prime number theorem states that the ratio $\frac{\pi(x)\log x}{x}$ approaches the limit 1 as x grows indefinitely large i.e.,

$$\lim_{x\to\infty}\frac{\pi(x)\log x}{x} = 1.$$

One usually writes $\pi(x) \sim \frac{x}{\log x}$ to describe such an asymptotic result. What Chebychev proved was that there are some explicit positive constants $a; b$ so that

$$a\frac{\log x}{x} < \pi(x) < b\frac{\log x}{x}.$$

If p_n denotes the n-th prime, Bertrand's postulate is equivalent to the assertion that $p_{n+1} < 2p_n$ while the prime number theorem itself is equivalent to the statement $\lim_{n\to\infty}\frac{p_n}{n\log n} = 1$.

If p_n denotes the n-th prime, Bertrand's postulate is equivalent to the assertion that $p_{n+1} < 2p_n$ while the prime number theorem itself is equivalent to the statement $\lim_{n\to\infty} (p_n)/(n \log n) = 1$.

It is rather startling to note that the great mathematician Gauss (1777-1855) had, at the age of 15, already conjectured the truth of the prime number theorem. Four years later, in 1796, Legendre also came independently to conjecture something similar.

Legendre conjectured, based on empirical evidence, that $\pi(x) \sim \frac{x}{A\log x+B}$ and also conjectured values of A, B which turned out to be incorrect. Gauss, on the other hand, conjectured that $\pi(x) \sim \int_2^x \frac{dt}{\log t}$; the right side is denoted by $li(x)$ to stand for the 'logarithmic integral' This seems to have exactly the content of the prime number theorem since clearly $li(x) \sim \frac{x}{\log x}$. However, later research (following Riemann) has confirmed that Gauss's assertion is even more astute than what it appears to be on the face of it. In fact, the function $li(x)$ has the asymptotic expansion (for any fixed n)

$$li(x) = \frac{x}{\log x} + 1!\frac{x}{(\log x)^2} + 2!\frac{x}{(\log x)^3} + \quad + (n-1)!\frac{x}{(\log x)^n} + O(\frac{x}{(\log x)^{n+1}}).$$

A refined version of the prime number theorem indeed implies that $\pi(x)$ has the same asymptotic expansion!

In particular, this implies that the best possible values for A and B in Legendre's conjecture are $A = 1, B = -1$.

The prime number theorem was proved independently by Hadamard and de la Vallee Poussin. A well-known mathematician quipped once that the proof almost immortalised these two mathematicians – they lived to be 96 and 98, respectively!

Returning to Bertrand's postulate, after Chebychev's first proof, other simpler proofs appeared. In this article, we shall discuss two of the simplest proofs due to

Ramanujan's solution of an elementary problem has been discussed in [1].

two great minds – Ramanujan and Erdos. It is perhaps for the first time that, in Resonance, a proof due to Ramanujan is given [1]. Most of us are told stories about him and his discoveries and it is rarely that one can find a proof of his which is elementary enough to be actually discussed at this level. Erdos's proof is even more elementary and we start with it.

Erdos's Proof

We start with any natural number n and look at the product $\prod_{p\leq n} p$ over all primes $p \leq n$.

We shall have occasion to use the well-known and easily proved fact asserting that the highest power to which a prime divides $n!$ is given by the expression

$$[n/p] + [n/p^2] + [n/p^3] +$$

Erdos's proof starts with the following very beautiful observation:

Lemma. $\prod_{p\leq n} p \leq 4^n$.

Proof.

The prime number theorem was proved independently by Hadamard and de la Vallee Poussin. A well-known mathematician quipped once that the proof almost immortalised these two mathematicians – they lived to be 96 and 98, respectively!

We prove this by induction on n. It evidently holds good for small n. Look at some $n > 1$ such that the result is assumed for all $m \leq n$. Then,

$$\prod_{p\leq n} p = \prod_{2p\leq n+1} p \prod_{n+1<2p\leq 2n} p$$

$$\leq 4^{\frac{n+1}{2}} \prod_{n+1<2p\leq 2n} p$$

by the induction hypothesis.

Now, the surprisingly simple observation that each prime in the last product (i.e., each prime between $(n+1)/2$ and n) divides the binomial coefficient $\binom{n}{[\frac{n+1}{2}]}$, shows that $\prod_{p\leq n} p \leq 4^{(n+1)/2} 2^{n-1} = 4^n$.

The bound $\binom{n}{[\frac{n+1}{2}]} \leq 2^{n-1}$ used above is trivially seen to be true by induction. Thus, we have proved this lemma.

Since we are interested in the possible primes between n and $2n$, it is natural to consider the binomial coefficient $\binom{2n}{n}$ because it 'captures' all these primes as its divisors. Now, obviously, the binomial coefficient $\binom{2n}{n}$ is the largest term in the expansion $(1+1)^{2n}$ which has $2n+1$ terms. Therefore, we have

$$(2n+1)\binom{2n}{n} \geq 2^{2n} \qquad (1)$$

This gives a lower bound for this middle binomial coefficient and, the idea of the rest of the proof is that lot of the contribution comes from primes between n and $2n$. More precisely, if we write

$$\binom{2n}{n} = \prod_{p\leq 2n} p^{e_p} = \prod_{p\leq n} p^{e_p} \prod_{n+1<p\leq 2n} p,$$

then we shall use the lemma to give an upper bound for the first product $\prod_{p\leq n} p^{e_p}$. Here, e_p denotes the power of p dividing the middle binomial coefficient $\binom{2n}{n}$ and, the second product stands for 1 if there are no terms.

We want to see which primes $p \leq n$ actually contribute to $\binom{2n}{n}$.

If $p^2 > 2n$ i.e., if $n \geq p > \sqrt{2n}$, then clearly, $e_p = [\frac{2n}{p}] - 2[\frac{n}{p}] = 0$ or 1.

Thus, such primes divide $\binom{2n}{n}$ either to a single power or not at all.

If $n \geq 3$, then a prime $p \leq n$ with $2n/3 < p$ must be at least 3 and so $p^2 \geq 3p > 2n$. As $1 \leq n/p < 3/2$ for $2n/3 < p \leq n$, we have $[\frac{n}{p}] = 1$ and $[\frac{2n}{p}] = 2$ i.e.,

$e_p = [\frac{2n}{p}] - 2[\frac{n}{p}] = 2 - 2 = 0$. Thus, these primes do not divide $\binom{2n}{n}$ when $n \geq 3$.

Gauss conjectured that $\pi(x) \sim \int_2^x (dt)/(\log t)$.

In other words, we have:

$e_p \le 1$ if $\sqrt{2n} < p \le 2n/3$, and $e_p = 0$ if $2n/3 < p \le n$.

Finally, for the primes with $p \le \sqrt{2n}$, we simply take the trivial bound $p^{e_p} \le 2n$. Then, we have

$$\binom{2n}{n} = \prod_{p \le n} p^{e_p} \prod_{n+1<p\le 2n} p$$

$$\le \prod_{p\le\sqrt{2n}} (2n) \prod_{\sqrt{2n}<p\le 2n/3} p \prod_{n+1<p\le 2n} p$$

$$\le 4^{2n/3} \prod_{p\le\sqrt{2n}} (2n) \prod_{n+1<p\le 2n} p.$$

using the lemma. We take $n \ge 8$ as we have verified Bertrand's postulate explicitly for $n \le 7$; so $\sqrt{2n} \ge 4$ and thus, the number of terms in the first product is at most $\sqrt{2n} - 2$ (as 1 and 4 are not primes). Therefore, we have on using (1) that

$$\frac{2^{2n}}{2n+1} \le \binom{2n}{n} \le (2n)^{\sqrt{2n}-2} 4^{2n/3} \prod_{n+1<p\le 2n} p.$$

Replacing the first term by $\frac{2^{2n}}{(2n)^2}$, we get

$$\prod_{n+1<p\le 2n} p \ge \frac{4^{n/3}}{(2n)^{\sqrt{2n}}}.$$

Thus, to show that the left side has terms (i.e., that it is not 1 according to our convention), it suffices to see whether the right hand side is bigger than 1 for all n. As usual, this will turn out to be true for large enough values of n and will fail for small values (this only means that the inequality is good enough for large values of n and we need to verify the original assertion directly for the smaller values left out).

After a few trials, we arrive at the number $n = 450$ and find that

$4^{n/3} = 4^{150} > (2n)^{\sqrt{2n}} = (900)^{30}$ since $4^5 > 900$.

There is nothing special about 450 excepting the fact that $2n$ is a perfect square and 450 is large enough for the inequality to hold good. This inequality continues to hold for $n > 450$ as the difference $4^{n/3} - (2n)^{\sqrt{2n}}$ is an increasing function. This last statement is simple to see by looking at the derivative of the difference of the corresponding logarithms. Now it is an easy exercise to verify Bertrand's postulate for $n < 450$. This was essentially Erdos's proof; ours is a slightly simplified version of his original argument which appears in [2].

Ramanujan's Proof

Let us turn to Ramanujan's proof. It is also extremely clever and completely elementary apart from the use of what is known as Stirling's formula – a proof of which has been discussed in [3].

We shall, however, give a further simplified version which avoids Stirling's formula altogether.

In the previous proof we used an estimate for $\prod_{p \leq n} p$. Here, we consider an additive version of it viz., look at the so-called Chebychev function $\theta(x) = \sum_{p \leq x} \log p$ defined for any real number $x \geq 2$. We note two things to begin with:

(i) $\theta(n)$ is simply the logarithm of $\prod_{p \leq n} p$,

(ii) Bertrand's postulate is true for a real number x if it is true for $n = [x]$; indeed, a prime between n and $2n$ is between x and $2x$ as well.

Let us also understand that since we are interested in primes between x and $2x$, we need a lower bound for $\theta(2x) - \theta(x)$. In other words, we need reasonable lower as well as upper bounds for θ values.

Now, the expression $\sum_{i \geq 1} [n/p^i]$ for the power of a prime dividing $n!$ gives us

$$\log[x]! = \sum_{i\geq 1} \Psi(x/i),$$

where Ψ is the function defined by

$$\Psi(x) = \sum_{i\geq 1} \theta(x^{1/i}).$$

This is the reason to introduce real x.

Using an elementary trick of old vintage, we have the following:

$$\log[x]! - 2\log[x/2]! = \sum_{i\geq 1}(-1)^{i-1}\Psi(x/i),$$

$$\Psi(x) - 2\Psi(\sqrt{x}) = \sum_{i\geq 1}(-1)^{i-1}\theta(x^{1/i}).$$

As θ, Ψ are increasing functions, we get inequalities by chopping off at an odd stage and at an even stage as follows:

$$\Psi(x) - \Psi(\frac{x}{2}) \leq \log[x]! - 2\log[\frac{x}{2}]!$$

$$\leq \Psi(x) - \Psi(\frac{x}{2}) + \Psi(\frac{x}{3}), \qquad (2)$$

$$\Psi(x) - 2\Psi(\sqrt{x}) \leq \theta(x) \leq \Psi(x). \qquad (3)$$

Here, Ramanujan takes recourse to Stirling's formula which states that $n! \sim \sqrt{2\pi}\frac{n^{n+\frac{1}{2}}}{e^n}$.

We shall not use it but proceed as follows.

For any $x > 1$, we have the binomial coefficient

$$\binom{[x]}{[\frac{x}{2}]} < \sum_{r\geq 0} \binom{[x]}{r} = 2^{[x]}.$$

Taking logarithms, we obtain

$$\log[x]! - 2\log[x/2]! < x\log 2.$$

But, clearly $\log 2 < \frac{3}{4}$ since $16 < e^3$. In other words, we have

$$\log[x]! - 2\log[x/2]! < 3x/4 \ \forall \ x > 0. \qquad (4)$$

Now, we find a lower bound. As we observed before Erdos's proof, for $x > 0$, the binomial coefficient $\binom{[x]}{[x/2]}$ (being the largest term in the expansion of $(1+1)^{[x]}$), must be bigger than $\frac{2^{[x]}}{[x]+1}$ since there are $[x]+1$ terms in the binomial expansion of $(1+1)^{[x]}$.

If x is large enough (for instance, if $x > 240$), then $\frac{2^{[x]}}{[x]+1} > e^{\frac{2[x]}{3}}$ Taking logarithms, we get

$$\log[x]! - 2\log[x/2]! > 2x/3 \ \forall \ x > 240. \qquad (5)$$

By (2),(4) and (5), we have

$$\Psi(x) - \Psi(x/2) < 3x/4 \ \forall \ x > 0, \qquad (6)$$

$$\Psi(x) - \Psi(x/2) + \Psi(x/3) > 2x/3 \ \forall \ x > 240. \qquad (7)$$

By replacing x by $x/2, x/4, x/8$ etc. in (6) and adding all the expressions we get

$$\Psi(x) < 3x/2 \ \forall \ x > 0. \qquad (8)$$

Note that since $\theta(x) < \Psi(x)$, we have a reasonable upper bound for $\theta(x)$ by (8). For the lower bound, let us use the first inequality of (3) viz., $\Psi(x) \leq 2\Psi(\sqrt{x}) + \theta(x)$ and the inequality $\theta(x/2) \leq \Psi(x/2)$ to write

$$\Psi(x) - \Psi(x/2) + \Psi(x/3) \leq 2\Psi(\sqrt{x}) + \theta(x) - \theta(x/2) + \Psi(x/3).$$

If we use the upper bound for Ψ given in (8), we obtain

$$\Psi(x) - \Psi(\frac{x}{2}) + \Psi(\frac{x}{3}) < \theta(x) - \theta(\frac{x}{2}) + \frac{x}{2} + 3\sqrt{x}. \qquad (9)$$

For $x > 240$, the left side has a lower bound given in (7) so that we finally obtain

$$\theta(x) - \theta(x/2) > x/6 - 3\sqrt{x} \ \forall \ x > 240.$$

Evidently, the right side is positive if $x > 324$. Therefore, there is a prime between x and $2x$ if $x > 162$. For smaller values of x, we find primes explicitly as before. This finishes the beautiful proof due to Ramanujan.

More Comments on Primes

The above elementary methods and their modifications are sufficient to prove Chebychev's theorem viz., the assertion

$$\frac{1}{6}\frac{x}{\log x} < \pi(x) < 6\frac{x}{\log x}.$$

The reader is urged to try and use this to prove the following bound for the n-th prime:

$$\frac{1}{6}n\log n < p_n < 12(n\log n + n\log\frac{12}{e}).$$

This last inequality (in fact, the upper bound) shows easily that the series $\sum \frac{1}{p}$ over all primes diverges. How fast does it diverge?

Here is a proof showing that the divergence is at least as fast as $\log\log x$ i.e.,:

$\exists c > 0$ *such that* $\sum_{p \leq x} \frac{1}{p} \geq c\log\log x \ \forall x.$

To see this, given $x > 1$, let us look at the area under the curve $y = \frac{1}{x}$ between 1 and x. Evidently, this area $\int_1^x \frac{dx}{x} = \log x$ is less than $\sum_{n=1}^{[x]} \frac{1}{n}$. (draw a figure to see this).

So, if we consider the product $\prod_{p \leq x}(1 - \frac{1}{p})^{-1}$, then clearly, $\prod_{p \leq x}(1 - \frac{1}{p})^{-1} \geq \sum_{n=1}^{[x]} \frac{1}{n} \geq \log x$.

Now, $(1 - \frac{1}{p})^{-1} = (1 + \frac{1}{p})(1 + \frac{1}{p^2 - 1})$. Therefore, $\prod_{p \leq x}(1 + \frac{1}{p}) \geq a\log x$ where $a = \prod_p (1 + \frac{1}{p^2-1})^{-1}$. Using $e^x > 1 + x$ for all $x > 0$, we get $\prod_{p \leq x} e^{1/p} \geq \prod_{p \leq x}(1 + \frac{1}{p}) \geq a\log x$ so that $\sum_{p \leq x} \frac{1}{p} \geq c\log\log x$ for some $c > 0$.

This finishes the proof of the lower bound.

The more adventurous reader may like to use the Abel summation formula (see [4]) and prove that this lower bound is of the correct order i.e., one has the rather interesting statement:

$$\sum_{p \leq x} \frac{1}{p} \sim \text{loglog} x.$$

We end the note by giving a curious application of Bertrand's postulate to finding a recursive expression for primes which appeared in ([5], p.289).

We consider the function

$$f_n = \text{Sign}(\frac{2((n-1)!)}{n} - [\frac{2((n-1)!)}{n}]$$

defined for all $n \geq 3$. Here, the sign function is the function which takes the value 0 at 0 and the value $\frac{x}{|x|}$ for any $x \neq 0$. Clearly, $f_n = 1$ or 0 according as n is prime or composite. Now, by Bertrand's postulate, if $p_n \geq 3$, then p_{n+1} occurs as the first prime among $p_n + 2, p_n + 4, \quad , 2p_n - 1$. Therefore, (writing p_n as p for simplicity of notation),

$$p_{n+1} = (p+2)f_{p+2} + (p+4)f_{p+4}(1 - f_{p+2}) +$$
$$+(p+6)f_{p+6}(1 - f_{p+2})(1 - f_{p+4}) + \quad \cdot +$$
$$(2p-1)f_{2p-1}(1 - f_{p+2}) \quad (1 - f_{2p-3}).$$

Suggested Reading

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