Fractals and the Large-Scale Structure in the Universe

2. Is the Cosmological Principle Valid?

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In Part 1 of this article, we had introduced the reader to some basic concepts of fractals and some operational algorithms to characterize them. We had also pointed out and clarified some common misconceptions about fractals. In this part we apply the fractal concepts to large-scale structures in the Universe. In particular we discuss the claim by some investigators that the distribution of galaxies in the Universe is a fractal. This is contrary to the standard view that the Universe is homogeneous and isotropic on large scales. Also discussed is the controversy as to whether the matter distribution in the Universe is a fractal on large-scales.

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Introduction

One of the most interesting aspects of our Universe is that there are structures of different sizes like planets, stars, star clusters, galaxies, clusters of galaxies, etc. Of these structures, the ones smaller than galaxies are believed to have been formed due to a number of physical processes in addition to gravity. Structures of galactic size and larger are supposed to have been formed predominantly due to gravitational condensation of matter in the Universe. The latter structures are broadly referred to as large-scale structures and understanding the physics operating at these scales is an important challenge to present day cosmology.

The standard model of cosmology rests on the assumption that although there are structures at various scales,

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the Universe is on the whole homogeneous and the structures are present within this homogeneous background. An alternative way of looking at it is that if we consider large enough scales, the Universe appears homogeneous and isotropic. Such an assumption (called the cosmological principle) has two main motivations at a theoretical level.

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1. One can argue from a Copernican point of view that one point in the Universe is no different from another. In other words, if the background Universe were to have an intrinsic inhomogeneity, some regions in the Universe would need a preferential treatment. This does not sound appealing. In a homogeneous Universe, however, there is no preferential treatment of any point and gross features in the Universe may depend on time but not on spatial coordinates.
2. From the point of view of mathematical convenience, the assumption of homogeneity and isotropy helps in simplifying the equations governing evolution of the background Universe.

However, an entirely different question is whether or not there is observational evidence for this homogeneity and isotropy. In other words, does the cosmological principle have compelling observational support.

In fact there are suggestions that the matter distribution in the Universe may have a fractal like characteristic.

Till recently the surveys of galaxies were not far enough in space to decide this question one way or the other. Observations have now started shedding a little light on this. However, the situation is far from clear and we would require deeper observations to answer this either way. In fact there are suggestions that the matter distribution in the Universe may have a fractal like characteristic. We first discuss what are fractals and what their salient features are. Then we turn to the question regarding the possible fractal nature of large-scale structures. That galactic distribution has fractal-like characteristics on small scales (< 5 Mpc) is a settled fact. The controversy is about whether there is a turnover to homogeneity on larger scales and if so at what scales.

Fractal Structure of the Universe

The standard model of cosmology is based on the assumption of homogeneity and isotropy of the Universe on very large scales. There are two questions which arise out of this.

1. Is this assumption correct ?
2. How large is 'large' ?

Since we cannot see the entire Universe, we can only ask, if there is evidence of homogeneity within the observed range of the Universe. If the galaxies are distributed homogeneously in three Euclidean dimensions, the number of galaxies $N(r)$ within a sphere of radius r should be proportional to r^3. However, if $N(r)$ is found to be proportional to r^D (where D is the fractal dimension) we can say that the distribution of galaxies is a fractal distribution in the observed range.

Determining the dependence of $N(r)$ on r is not as straightforward as it may appear. Before examining this issue in detail, it is necessary to become aware of certain aspects of our knowledge of galaxy distribution. To obtain a full picture of the galactic distribution in the Universe, ideally one should have a complete knowledge about the position of all galaxies in a given volume V Whereas the direction (angular position) of a galaxy is easy to determine, the determination of the radial distance of the galaxy from us is more complicated. The redshift of the galaxies are a measure of their distance from us. However, determining the distance from the redshift data, involves, several correction factors. We

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will not discuss these correction schemes here, as it will be a substantial digression from the aims of this article.

Interpreting galaxy catalogs have several problems arising out of observational constraint. It has to be realized that there is always an observational threshold. So the galaxies of apparent luminosity below this threshold value are not included in a catalog. This would introduce a bias leading to observation of fewer galaxies at larger distances. To overcome this bias, one constructs a *volume limited* catalog by fixing a certain depth and including in the distribution only those galaxies whose absolute luminosity is large enough that it would be visible at this sample depth. In this way the distribution cuts off all galaxies below a threshold absolute luminosity instead of excluding all galaxies below a threshold apparent luminosity. This leads to a considerably reduced sample available for statistical analyses.

Having, determined the position of galaxies in the catalog, we need to do its statistical analysis The conventional analysis of galaxy distributions is based on the notion of correlation length.

To understand the notion of correlation length, consider a typical liquid. Let $n(\vec{r})$ denote the number density of molecules of the liquid at a point labeled by the position vector $\vec{r}$. In a homogeneous liquid $n(\vec{r})$ is independent of $\vec{r}$. However, if a molecule is known to be present at a position $\vec{r}_0$ then another molecule cannot be present very close to it. Notice that the function $\langle n(\vec{r}_0)n(\vec{r}_0 + \vec{r})n\rangle$ averaged over all positions $\vec{r}_0$ is small, when r is small compared to inter-molecular distances; grows to a maximum when r is of the order of inter-molecular separation; oscillates for a while and finally settles at a value $\langle n\rangle^2$ for r much larger than the inter-molecular separation. It is natural to divide this function by $\langle n\rangle^2$ to obtain a dimensionless quantity and then subtract 1 so that the saturation value is 0. In this way one obtains the two-point correlation function ξ defined by,

$$\xi(\vec{r}) = \frac{\langle n(\vec{r}_0)n(\vec{r}_0 + \vec{r})\rangle}{\langle n\rangle^2} - 1. \quad (1)$$

One can pass from the vector variable $\vec{r}$ to a scalar one by taking an angular average over all possible directions of $\vec{r}$.

Above a certain length scale r, called the correlation length, the correlation function becomes negligibly small. Correlation length can obviously be interpreted as the scale at which the liquid appears homogeneous. It should be noted however that the above definition is meaningful only if $\langle n\rangle$ is well defined in the sense that it is independent of the sample size. In other words, the correlation length obtained for water in a glass should be the same for that obtained for water in a lake, if the correlation length is to define a property of water and not the container.

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In galaxy distributions, however, having a well defined $\langle n\rangle$ in the above sense is a nontrivial issue. This is the primary source of controversy about the nature of galaxy distributions.

For the galactic distribution for small values of r the correlation function was found to obey a power law $\xi(r) \sim r^{3-D}$ In the conventional analysis of galaxy distributions, correlation length r_0 was defined as the value for which the correlation function $\xi(r_0) = 1$. Based on this analysis the galaxy correlation length was determined to be about 5 Mpc. Beyond this value $\xi(r)$ was found to rapidly decay to zero. This was interpreted to mean that galaxies are strongly correlated at small distances, while for $r > r_0$ the correlation decays rapidly and the galaxy distribution becomes homogeneous. Similarly an analysis of cluster distributions led to a cluster correlation length of 25 Mpc. The problem with these methods of analysis is that they assume a priori that the system

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is homogeneous at some large-scale, but if the homogeneity assumption is not valid, they lead to spurious results.

Use of the above mentioned correlation function for analyzing the large-scale structure of the Universe will be valid only if the size of the sample of the Universe being analyzed is large enough to yield a unique value of $\langle n \rangle$. If this is not the case, the correlation length obtained will be spurious, as it will depend on the size of the portion of the Universe being analyzed.

This is best illustrated by applying the conventional correlation analysis on a model fractal Universe [1]. A model fractal Universe shows spurious correlation lengths, even though it is known that there is no length above which the model Universe becomes homogeneous. This shows that the conventional correlation analysis is unable to distinguish between a homogeneous Universe and a fractal Universe if the samples are not deep enough.

For a fractal of dimension D, the number of galaxies in a sphere of radius r, centered on the galaxy, is given by $N(r) \sim r^D$ If the sample size is denoted by R_s, the average density over the sampled space is given by $\langle n \rangle \sim R_s^{D-3}$. The value of $\langle n(\vec{r}_0)\ n(\vec{r}_0 + \vec{r})\rangle$ will be $\sim \langle n \rangle r^D$ Clearly the correlation function as defined above will depend on the sample size R_s. However, the conditional density function defined by $G(\vec{r}) = \langle n(\vec{r}_0) n(\vec{r}_0 + \vec{r})\rangle / \langle n \rangle$ will be independent of the sample size. The conventional use of the correlation function led to a spurious value that depends on the sample size, whereas the use of G gives no evidence of turnover to homogeneity and is of power-law type. If $G(r) = Ar^g$ then $G(br) = A(br)^g = Ab^g r^g = Ab^g G(r)$. Scaling r by a constant factor b, scales the correlation function by the constant b^g. This shows that power-law scaling is indicative of a self-similar fractal structure.

One can also create a model Universe, which is fractal up to a scale l_0, but is homogeneous on larger scales. The analysis of this Universe also shows that the correlation length obtained by using the function ξ will depend on both l_0 and the sample size. Unless the sample size is large compared to l_0, this correlation length would be spurious. In contrast the conditional density function G, as expected, is found to be independent of the sample size. It is a power law up to l_0 and a constant beyond that. The length scale at which G turns from a power law to a constant value gives the true correlation length.

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It is clear from the above analysis on model fractal Universes, that the correlation studies should use the conditional density function G instead of the correlation function ξ in order to obtain the intrinsic character of the correlation properties independent of the sample size. Some workers have found evidence for turnover from fractal to homogeneity on scales of $100-200$ Mpc. However, deeper catalogs will be required to enable one to decide unambiguously the now hotly debated question of whether the Universe continues to be a fractal on very large scales or whether there is a scale above which the Universe becomes homogeneous.

One point that needs to be emphasized, lest it be overlooked, is that the fractal-scaling behavior holds for any arbitrary galaxy treated as the origin. *A fractal Universe is not homogeneous in the conventional sense of uniform number density; but it is homogeneous in the sense that the same fractal scaling behavior is observed from any galaxy, that is from an occupied point of the fractal.* A homogeneous Universe appears to be the same from every point in space. This is the cosmological principle. The fractal Universe appears to be the same only from occupied points. This is called the conditional cosmological principle. Recently there has been an attempt to construct a dynamical fractal model satisfying the conditional cosmological principle and the general theory of relativity (see [2, 3]).

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Chaos and Fractals

It is not immediately obvious why fractals should play an important role in description of nature. Nature is overwhelmingly dominated by dynamical systems whose evolution depends sensitively on initial conditions. Such systems are called chaotic dynamical systems. Before the understanding of chaotic systems emerged, it was believed that bounded motion of any dynamical system would eventually approach a fixed point, a periodic orbit or a quasi-periodic orbit. However, *chaotic systems eventually approach more complicated sets called strange attractors.* These sets are usually fractals. This explains the preponderance of fractals in Nature.

Fractals in General Relativity and the Early Universe

Standard models of the Early Universe reduce the early Universe cosmology into a relatively simple integrable model by making simplistic assumptions. More realistic assumptions lead to complex models displaying chaotic evolution. Chaos in Newtonian dynamics is characterized by positivity of Lyapunov exponents, numbers that measure the average rate of separation of nearby trajectories. If $e(t)$, the distance between two neighboring trajectories at time t is given by $e(t) = e_0 e^{\lambda t}$, then λ is called the Lyapunov exponent. If $\lambda > 0$, then the separation between neighboring trajectories grows exponentially and the dynamical system exhibits sensitive dependence on initial conditions. For general relativistic dynamical systems any characteristic of chaos must be invariant under arbitrary differentiable transformation of coordinates. The Lyapunov exponent does not meet this requirement. For example, the transformation $t \to \ln(t)$ will transform $e^{\lambda t}$ to t^{λ}. Thus what will appear to be chaotic if coordinate t is used, will appear non-chaotic if coordinate $\ln(t)$ is used. More fundamentally, the definition of Lyapunov exponent puts the space and time coordinates on distinct footing, whereas General Relativity puts space and time coordinates on equal footing. The concept of Lyapunov exponent is therefore not suitable for characterizing chaos in curved space-time of the early Universe, which is described by general relativity.

In contrast, *fractals provide an invariant descriptor of chaos*. No smooth map can convert a fractal into a non-fractal, as fractals are non-differentiable. Moreover, the dimension of a fractal is a coordinate independent topological feature. This fact makes fractals very useful in characterizing chaotic features in general relativity.

The set of initial conditions that lead to a particular cosmological outcome is called a basin of attraction for that particular outcome. For integrable models the basins of attraction for different outcomes are separated by smooth, regular boundaries. For chaotic evolution, these boundaries break up and become fractal. Moreover, chaotic cosmological evolution can lead to a fractal pattern in the spectrum of density fluctuations.

In conclusion, whether or not the Universe has a fractal distribution of matter is not very clear. We now have very deep surveys. But these are not yet deep enough to make a definitive statement. With the available observations, there seem to be some indications that there is a transition to homogeneity at a scale of about 100 Mpc [4]. But it appears that although we have at our disposal deep surveys, they are just deep enough to give some hints about the transition. To make a decisive statement we need observations which are still deeper. On the theoretical side, the likelihood of cosmic evolution being governed by a chaotic dynamical system indicates that the Universe could have a fractal structure.

Suggested Reading

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