In this section of Resonance, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Sums of Powers of the Primitive Roots of a Prime

In [1], the construction of regular polygons by a ruler and a compass is discussed. In the last section of the article, the notion of cyclotomic polynomials is employed to evaluate the sum of the primitive roots of a prime $p$. This turns out to be $\mu(p - 1)$ where $\mu$ is the Möbius function. The general question of evaluating the sum of the $m$-th powers of the primitive roots is also raised. Here, we answer this question in an elementary manner. Recall that a natural number $a$ is a primitive root of a prime $p$ if $p - 1$ is the smallest natural number for which $a^{p-1} \equiv 1 \pmod{p}$. Let $1 \leq r_1, r_2, \ldots, r_k \leq p - 1$ be the integers that are co-prime to $p - 1$. Then if $w$ is a primitive root of $p$, we know that $w^{r_1}, w^{r_2}, \ldots, w^{r_k}$ are all the primitive roots.

We wish to evaluate the sum $S = \sum_{i=1}^{k} (w^{r_i})^m$. Let us note that as primitive roots are defined only modulo $p$, this sum will be evaluated only modulo $p$. 

Here and elsewhere in this proof, we write \( a = b \) to mean \( a \equiv b \mod p \). Thus \( S \) is simply the congruence class modulo \( p \) to which \( \sum_{i=1}^{k} (w^{r_i})^m \) belongs.

Let us start with the useful observation (here and elsewhere \((a,b)\) denotes the GCD of two natural numbers):

**Lemma.** For an integer \( q \), let \((p-1,q) = d\). Then, if \( t \) divides \( p - 1 \),

\[
\sum_{\ell=1}^{(p-1)/t} w^{tq\ell} = \begin{cases} 
0 & \text{if } \frac{p-1}{d} \not| t \\
\frac{p-1}{t} & \text{if } \frac{p-1}{d} | t
\end{cases}
\]

**Proof.**

\[
w^{tq} = 1 \Leftrightarrow p - 1 | tq \Leftrightarrow \frac{p - 1}{d} | t
\]

In this case \( \sum_{\ell=1}^{(p-1)/t} w^{tq\ell} = 1 + 1 + \cdots + 1 = \frac{p-1}{t} \).

If \( w^{tq} \neq 1 \), then

\[
\sum_{\ell=1}^{(p-1)/t} w^{tq\ell} = w^{tq} + w^{2tq} + \cdots + w^{(p-1)q}
\]

\[
= \frac{w^{tq}(w^{(tq)\frac{p-1}{t}} - 1)}{w^{tq} - 1}
\]

\[
= 0.
\]

We shall prove:

**Theorem.** The sum \( S \) of \( m \)-th powers of primitive roots for \( p \) is given by \( S = \mu(g) \frac{\phi(p-1)}{\phi(g)} \) where \( g = \frac{p-1}{(m,p-1)} \).

Here \( \phi \) and \( \mu \) are Euler’s phi function and the Möbius function respectively. We shall evaluate \( S \) by using the inclusion-exclusion principle.

**Proof.** Let \( p_1, p_2, \ldots, p_s \) be the various distinct prime divisors of \( p - 1 \). Thus

\[
S = \sum_{i=1}^{k} (w^{r_i})^m = \sum_{i=1}^{p-1} w^{im} - \sum_{j=1}^{s} \sum_{i=1}^{p-1} (w^{ip_j})^m
\]
The above equality is deduced as follows. Let $T = \{1, 2, \ldots, p - 1\}$ and let $T_f$ denote the subset of $T$ consisting of those integers from $T$ which are divisible by $f$. Then by the inclusion-exclusion principle, one gets:

$$S = \sum_{(x,p-1)=1} (w^x)^m - \sum_{x \in T} (w^x)^m + \sum_{x \in T_{p_1}} (w^x)^m + \sum_{x \in T_{p_2}} (w^x)^m - \sum_{i<j} (w^x)^m + (-1)^s \sum_{x \in (T_{p_1} \cap T_{p_2} \cap \ldots \cap T_{p_s})} (w^x)^m$$

Finally, as it is clear that

$$\sum_{x \in (T_{p_1} \cap T_{p_2} \cap \ldots \cap T_{p_s})} (w^x)^m = \sum_{i=1} (w^{ip_{j_1}p_{j_2} \ldots p_{j_u}})^m$$

we obtain the expression $\spadesuit$ for $S$.

Now $\{p_1, p_2, \ldots, p_s\}$ is the set of all prime divisors of $p - 1$. Consider its subset $\{p_1, p_2, \ldots, p_t\}$, the set of prime divisors of $g = \frac{p-1}{(m,p-1)}$. Then, by the lemma, a sum of the form $\sum_{i=1} (w^{ip_{j_1}p_{j_2} \ldots p_{j_u}})^m$ is not equal to 0 if and only if $g | p_{j_1}p_{j_2} \ldots p_{j_k}$. Clearly this happens only if $g$ is squarefree. Assume $g$ is squarefree; then $g = p_1p_2 \ldots p_t$. So, in evaluating $S$, we only have to find the sum of all terms of the form

$$(-1)^u \sum_{i=1} (w^{ip_{j_1}p_{j_2} \ldots p_{j_u}})^m$$
where \( \{1, 2, \ldots, t\} \subseteq \{j_1, j_2, \ldots, j_u\} \). But, the lemma gives us

\[
\sum_{i=1}^{(p-1)/(p_j_1 p_j_2 \cdots p_j_u)} (w_i p_j_1 p_j_2 \cdots p_j_u)^{m} = \frac{p - 1}{p_j_1 p_j_2 \cdots p_j_u}
\]

whenever \( \{1, 2, \ldots, t\} \subseteq \{j_1, j_2, \ldots, j_u\} \). Hence, we have

\[
S = (-1)^t \frac{p - 1}{p_1 p_2 \cdots p_t} + \\
(-1)^{t+1} \left[ \frac{p - 1}{p_1 p_2 \cdots p_t p_{t+1}} + \frac{p - 1}{p_1 p_2 \cdots p_t p_s} \right] + (-1)^{t+2} \left[ \frac{p - 1}{p_1 p_2 \cdots p_t} \left( \frac{1}{p_{t+1} p_{t+2}} + \frac{1}{p_{t+1} p_{t+3}} + \frac{1}{p_{s-1} p_s} \right) \right] \pm (-1)^s \frac{p - 1}{p_1 p_2 \cdots p_s}
\]

\[
= (-1)^t \frac{p - 1}{p_1 p_2 \cdots p_t} \left[ 1 - \left( \frac{1}{p_{t+1}} + \frac{1}{p_s} \right) \right] + \left( \frac{1}{p_{t+1} p_{t+2}} + \frac{1}{p_{s-1} p_s} \right) + \frac{(-1)^{s-t}}{p_{t+1} p_{t+2} \cdots p_s}
\]

\[
= (-1)^t \left( \frac{p - 1}{p_1 p_2 \cdots p_t} \right) \left( \frac{1 - \frac{1}{p_{t+1}}}{1 - \frac{1}{p_1}} \right) \left( \frac{1 - \frac{1}{p_2}}{1 - \frac{1}{p_2}} \right) \left( \frac{1 - \frac{1}{p_s}}{1 - \frac{1}{p_s}} \right) \left( \frac{1 - \frac{1}{p_{t+2}}}{1 - \frac{1}{p_{t+2}}} \right) \left( \frac{1 - \frac{1}{p_{s-1}}}{1 - \frac{1}{p_{s-1}}} \right)
\]

\[
= (-1)^t \left( \frac{p - 1}{p_1 p_2 \cdots p_t} \right) \phi(p - 1)/(p - 1) = \mu(g) \frac{\phi(p - 1)}{\phi(g)}
\]

since \( g = p_1 \cdots p_t \) and \( (-1)^t = \mu(g) \).

Thus whenever \( g \) is squarefree, \( S = \frac{\mu(g) \phi(p - 1)}{\phi(g)} \). But, if \( g \) is not squarefree, \( g \) cannot divide \( p_j_1 p_j_2 \cdots p_j_u \); so each term of \( \diamondsuit \) is 0 and \( S = 0 \). Also \( \mu(g) \frac{\phi(p - 1)}{\phi(g)} = 0 \) if \( g \) is not squarefree. Therefore, in all cases \( S = \mu(g) \frac{\phi(p - 1)}{\phi(g)} \).

Suggested Reading