K S Krishnan's 1948 Perception of the Sampling Theorem

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K S Krishnan discovered in 1948 that the sum over samples of a band-limited function gives the value of the integral of the function exactly, provided the sampling interval does not exceed a threshold value. This result of Krishnan is shown to be equivalent to the Shannon sampling theorem.

Let $f(x)$ be a function of the real variable x. The integral of $f(x)$ is the area under the graph of $f(x)$. We may divide this area into thin vertical strips of width α and height equal to the value of $f(x)$ at the location of the strip. Then the sum of the areas of these strips will approximate the true value of the integral if the spacing α is sufficiently small:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}\,x \approx \alpha \sum_{-\infty}^{\infty} f(n\alpha + x_0), \quad \text{for small} \quad \alpha$$

That is, the integral is approximated by a sum over the samples; $0 \leq x_0 < \alpha$ is the smallest nonnegative value of x at which a sample is located. One expects the value of the sum, and hence its departure from the value of the integral, to depend not only on α, but also on x_0. For instance, if $f(x) = \exp(-x)$ for $x \geq 0$, and $f(x) = 0$ for $x < 0$, the integral on the left has unit value but the sum on the right assumes the value

$$\alpha \sum_{-\infty}^{\infty} f(n\alpha + x_0) = \frac{\alpha\,\exp(-x_0)}{1 - \exp(-\alpha)}.$$

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Sampling theorem, K S Krishnan, Shannon.

One also expects the approximation of the integral by the sum to become better and better with decreasing

α, the sum yielding the (Riemann) integral itself in the limit $\alpha \to 0$

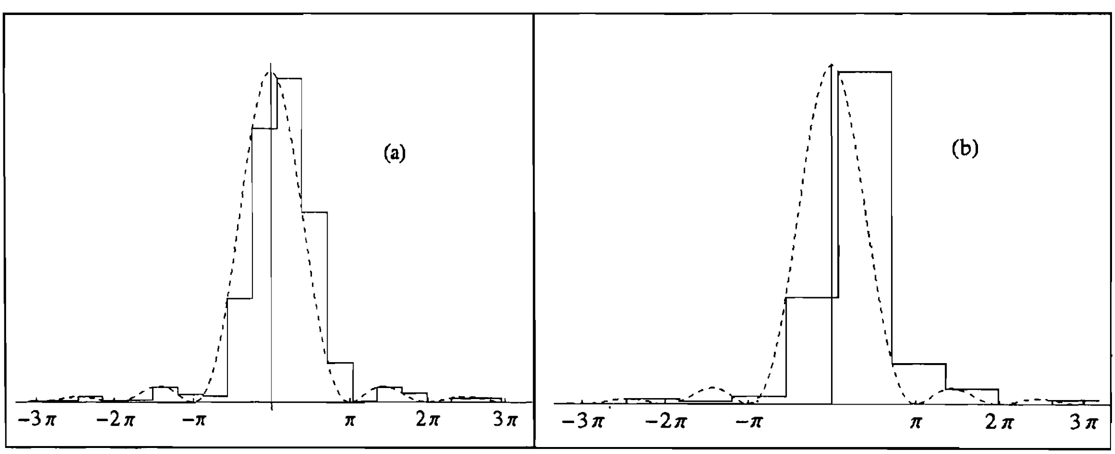
The sinc function defined as $\operatorname{sinc} y = \frac{\sin y}{y}$, and its square, appear frequently in applications in science and engineering. K S Krishnan found that the sinc-square function has a property which may appear bizarre from the viewpoint of the above consideration:

$$\alpha \sum_{n=-\infty}^{\infty} \frac{\sin^2(n\alpha + x_0)}{(n\alpha + x_0)^2} = \pi = \int_{-\infty}^{\infty} dx \, \frac{\sin^2 x}{x^2}, \quad \text{for } \alpha \leq \pi \text{ and any } x_0 . \tag{1}$$

That is, the sum equals the integral *exactly* not just for infinitesimal values but for any finite value of the sampling interval $\leq \pi$. *Figure* 1 shows plot of the sinc-square function, and a representation of the sum, for two values of α.

Krishnan is well known as one of the greatest experimental physicists this country has ever seen. What may not be widely known is the fact that Krishnan loved mathematics and those who knew him had great respect and even awe for his skill as a mathematician. Indeed, Lonsdale and Bhabha in their biographical notes on Krishnan record the following: "Krishnan loved mathematical reasoning and his skill as a mathematician would have

Figure 1. Illustrating the identity of the integral and sum in (1): (a) corresponds to α = 1, x_0 = 0.25 and (b) corresponds to α = 2, x_0 = 0.25. In either case the area under the dotted curve representing the integral in (1) equals, exactly, the area under the rectangulated version representing the sum.



gained him international recognition even without his great ability as an experimental physicist"

Krishnan was intrigued by the two *symmetries* the sum of the sinc-square samples possesses, namely invariance under translation (x_0) by arbitrary amount and scaling of the sampling interval (α) by a restricted amount. He set forth two questions for him to answer: Can the above symmetries be understood in terms of some generic property of the function under consideration? Are there other functions exhibiting these symmetries? It turns out that the two questions are related.

Krishnan recognized that his original proof of (1), based on the identity

$$\sum_{n=-\infty}^{\infty} \frac{\sin\,(n+\beta)z}{n+\beta} = \pi \quad \text{for any} \quad |z| < 2\pi \qquad (2)$$

was function-specific, and that it was therefore unlikely to help in identifying the generic property he was after. And so he considered a second proof, with help from Wiener which he acknowledges in a footnote: "We are thankful to Professor Norbert Wiener for the following elegant alternative proof"

The proof this time is based on Fourier transform. Let $f(x)$ be such that the Fourier transform

$$g(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) \exp(-ivx)$$

exists. The integral of $f(x)$ is nothing but the Fourier transform evaluated at frequency $v = 0$. It is the 'dc component' of $f(x)$. Krishnan assumes $f(x)$ to be real and even. Then $g(v)$ too will be real and even. Exploiting the Poisson's summation formula, Krishnan proves the result:

Krishnan's Theorem: If $f(x)$ is a real even function whose Fourier transform $g(v)$ vanishes for all $|v| \geq v_0$, then

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha + x_0) = \int_{-\infty}^{\infty} dx\, f(x) \quad \text{for every } 0 < \alpha \leq 2\pi/v_0 \quad (3)$$

independent of x_0. If $g(v) \neq 0$ at $v = \pm v_0$, then α is restricted by $0 < \alpha < 2\pi/v_0$.

That there exists a copious supply of band-limited functions is due to the fact that these functions have a knack of cross as well as self-breeding.

This answers, in one stroke, the two questions Krishnan set out to clarify: the generic property that Krishnan was after is one of *band-limitedness* [the function $f(x)$ is said to be band-limited if its Fourier transform $g(v)$ vanishes for all frequencies outside a band $v \leq v_0$; the parameter v_0 is called the *bandwidth* of $f(x)$]. Thus what appeared to be a strange property special to sinc-square function is shown to apply to *every* band-limited function. It turns out that *Krishnan's theorem holds also for functions which are neither real-valued nor even*; band-limitedness is the lone requirement for its validity! Band-limited functions form a large family, and there lies the importance of Krishnan's result: for every band-limited function we have an identity expressing the integral as a sum over the samples (and this can be handy if one of them is easy to evaluate but the other is not). We may note that the sinc function is band-limited, and the identity in (2) is one of this origin.

That there exists a copious supply of band-limited functions is due to the fact that these functions have a knack of cross as well as self-breeding: (i) A linear combination of two band-limited functions is band-limited, with bandwidth equal to the *larger* of the individual bandwidths; (ii) From the convolution property it is transparent that the product of two band-limited functions is band-limited, with bandwidth equal to the *sum* of the individual bandwidth; (iii) Since convolution translates under Fourier transformation into pointwise multiplication, it follows that the convolution of two band-limited

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functions results in a band-limited function whose bandwidth is the *smaller* of the individual bandwidths [We can make a stronger statement, for band-limited functions have the following 'ideal' property: convolution of a band-limited function with *any* function is band-limited!]; (iv) Derivative of a band-limited function is band-limited, with the same bandwidth; (v) Finally, if $f(x)$ is band-limited, then $f(x-a)$ and $f(x)\exp(i\,c\,x)$ for any real a, c, are band-limited as well.

Krishnan's result, with the reality and symmetry restrictions relaxed, is equivalent to what is popularly known as *Shannon sampling theorem*, which asserts that a band-limited function of bandwidth v_0 can be *fully* reconstructed from its samples if the sampling interval α is less than $(2v_0)^{-1}$. This may be seen from the following two observations. First, the Fourier transform relation in (3) is invertible, and hence knowing the spectrum $g(v)$ is equivalent to knowing $f(x)$ itself. Second, the Fourier transform $g(v)$ evaluated at v is simply the integral of the product function $f(x)\exp(-ivx)$, which is band-limited with bandwidth $\leq (v_0 + |v|)$, and thus the evaluation of the Fourier integral is governed by Krishnan's theorem. In a paper written in honour of Krishnan in the special issue of *Current Science* brought out on the occasion of his birth centenary, I have given a more careful exposition of this equivalence.

This work of Krishnan was published in 1948 when Krishnan was Director of the National Physical Laboratory, through a short note in *Nature*, titled 'A simple result in quadrature', followed by a detailed one in the *Journal of the Indian Mathematical Society*, under the title 'On the equivalence of certain infinite series and the corresponding integrals'. Interestingly, this is about the time Shannon's influential work on sampling theorem appeared. This reference to the point of time in history is not to imply that Krishnan's work marked the discovery of the sampling theorem. As a mathematical fact, the sampling theorem was known to some before Shannon who, indeed, acknowledges in his paper an early work of Whittaker. But it is the work of Shannon that established the sampling theorem as a powerful tool in the arsenal of electrical engineers. Krishnan's perception of the sampling theorem was quite different: he viewed it as a rich source of mathematical identities.

While reading Krishnan's work, I was struck by this famous experimental physicist's fascination for, and familiarity with, the work of 'the man who knew infinity'.

Suggested Reading

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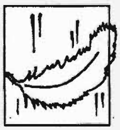

Trouble about common sense is that it can often let you down. After all, common sense suggests that the sun and stars revolve around the Earth. Einstein once remarked that *"common sense is that layer of prejudices laid down in the mind prior to the age of eighteen"* [quoted in *Relativity for Scientists and Engineers* by R Skinner (Dover, New York, 1982), p.27.].

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