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Eigenvalues of AB and BA

Let A, B be $n \times n$ matrices with complex entries. Given below are several proofs of the fact that AB and BA have the same eigenvalues. Each proof brings out a different viewpoint and may be presented at the appropriate time in a linear algebra course.

Let $\operatorname{tr}(T)$ stand for the trace of T and $\det(T)$ for the determinant of T The relations

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \quad \text{and} \quad \det(AB) = \det(BA). \tag{1}$$

are usually proved early in linear algebra courses. The first is easy to verify; the second takes more work to prove.

Let

$$\lambda^n - c_1(T)\lambda^{n-1} + \quad + (-1)^n c_n(T) \tag{2}$$

be the characteristic polynomial of T and let $\lambda_1(T)$, $\lambda_2(T)$, $\lambda_n(T)$ be its n roots, counted with multiplicities and in any order. These are the eigenvalues of T We know that $c_k(T)$ is the kth elementary symmetric polynomial in these numbers. Thus

$$\begin{aligned}
c_1(T) &= \sum_{j=1}^{n} \lambda_j(T) = \operatorname{tr}(T) \\
c_2(T) &= \sum_{i<j} \lambda_i(T)\lambda_j(T) \\
&\ \vdots \\
c_n(T) &= \prod_{j=1}^{n} \lambda_j(T) = \det(T).
\end{aligned}$$

To say that AB and BA have the same eigenvalues amounts to saying that

$$c_k(AB) = c_k(BA) \quad \text{for } 1 \le k \le n. \tag{3}$$

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Eigenvalues, idempotent, projection operator, spectrum, Hilbert space.

We know that this is true when $k = 1$, or n ; and want to prove it for other values of k.

Proof 1. It suffices to prove that, for $1 \leq m \leq n$,

$$\lambda_1^m(AB) + \cdots + \lambda_n^m(AB) = \lambda_1^m(BA) + \cdots + \lambda_n^m(BA) \quad (4)$$

(Recall Newton's identities by which the n elementary symmetric polynomials in n variables are expressed in terms of the n sums of powers.) Note that the eigenvalues of T^m are the mth powers of the eigenvalues of T So, $\sum \lambda_j^m(T) = \sum \lambda_j(T^m) = \text{tr } (T^m)$. Thus the statement (4) is equivalent to

$$\text{tr } [(AB)^m] = \text{tr } [(BA)^m].$$

But this follows from (1)

$$\begin{aligned} \text{tr } [(AB)^m] = \text{tr } (ABAB \cdots AB) &= \text{tr}(BABA \cdots BA) \\ &= \text{tr } [(BA)^m]. \end{aligned}$$

Recall Newton's identities by which the n elementary symmetric polynomials in n variables are expressed in terms of the n sums of powers.

Proof 2. One can prove the relations (3) directly. The coefficient $c_k(T)$ is the sum of all the $k \times k$ principal minors of T A direct computation (the Binet–Cauchy formula) leads to (3). A more sophisticated version of this argument involves the antisymmetric tensor product $\wedge^k(T)$. This is a matrix of order $\binom{n}{k}$ whose entries are the $k \times k$ minors of T So

$$c_k(T) = \text{tr } \wedge^k (T), 1 \leq k \leq n.$$

Among the pleasant properties of $\wedge^k$ is multiplicativity: $\wedge^k(AB) = \wedge^k(A) \wedge^k (B)$. So

$$\begin{aligned} c_k(AB) &= \text{tr } [\wedge^k(AB)] = \text{tr } [\wedge^k(A) \wedge^k (B)] \\ &= \text{tr } [\wedge^k(B) \wedge^k (A)] = \text{tr } \wedge^k (BA) = c_k(BA). \end{aligned}$$

Proof 3. This proof invokes a continuity argument that is useful in many contexts. Suppose A is invertible (nonsingular). Then $AB = A(BA)A^{-1}$. So AB and BA are

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similar, and hence have the same eigenvalues. Thus the equalities (3) are valid when A is invertible. Two facts are needed to get to the general case from here. (i) if A is singular, we can choose a sequence A_m of nonsingular matrices such that $A_m \to A$. (Singular matrices are characterised by the condition det $(A) = 0$. Since det is a polynomial function in the entries of A, the set of its zeros is small. See also the discussion in *Resonance*, Vol. 5, no. 6, p. 43, 2000). (ii) The functions $c_k(T)$ are polynomials in the entries of T and hence, are continuous. So, if A is singular we choose a sequence A_m of nonsingular matrices converging to A and note

$$c_k(AB) = \lim_{m\to\infty} c_k(A_mB) = \lim_{m\to\infty} c_k(BA_m) = c_k(BA).$$

Proof 4. This proof uses 2×2 block matrices. Consider the $(2n)\times(2n)$ matrix $\begin{bmatrix} X & Z \\ O & Y \end{bmatrix}$ in which the four entries are $n\times n$ matrices, and O is the null matrix. The eigenvalues of this matrix are the n eigenvalues of X together with the eigenvalues of Y (The determinant of this matrix is $\det(X)\det(Y)$.) Given any $n\times n$ matrix A, the $(2n)\times(2n)$ matrix $\begin{bmatrix} I & A \\ O & I \end{bmatrix}$ is invertible, and its inverse is $\begin{bmatrix} I & -A \\ O & I \end{bmatrix}$ Use this to see that

$$\begin{bmatrix} I & A \\ O & I \end{bmatrix}^{-1} \begin{bmatrix} AB & O \\ B & O \end{bmatrix} \begin{bmatrix} I & A \\ O & I \end{bmatrix} = \begin{bmatrix} O & O \\ B & BA \end{bmatrix}.$$

Thus the matrices $\begin{bmatrix} AB & O \\ B & O \end{bmatrix}$ and $\begin{bmatrix} O & O \\ B & BA \end{bmatrix}$ are similar and hence, have the same eigenvalues. So, AB and BA have the same eigenvalues.

Proof 5. Let A be an idempotent matrix, i.e., $A^2 = A$. Then A represents a projection operator (not necessarily an orthogonal projection). So, in some basis (not necessarily orthonormal) A can be written as $A = \begin{bmatrix} I & O \\ O & O \end{bmatrix}$ In this basis let $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ Then $AB = \begin{bmatrix} B_{11} & B_{12} \\ O & O \end{bmatrix}$ $BA = \begin{bmatrix} B_{11} & O \\ B_{21} & O \end{bmatrix}$. So, AB and BA have the same eigenvalues. Now let A be any matrix. Then there exists an invertible matrix G such that $AGA = A$. (The two sides are equal as operators on the null space of A. On the complement of this space, A can be inverted. Set G to be the identity on the null space of A.) Note that GA is idempotent and apply the special case to GA and BG^{-1} in place of A and B. This shows $GABG^{-1}$ and $BG^{-1}GA$ have the same eigenvalues. In other words AB and BA have the same eigenvalues.

A nonzero eigenvalue of AB has the same *geometric multiplicity* as it has as an eigenvalue of BA. This may not be true for a zero eigenvalue.

Proof 6. Since $\det AB = \det BA, 0$ is an eigenvalue of AB if and only if it is an eigenvalue of BA. Suppose a nonzero number λ is an eigenvalue of AB. Then there exists a (nonzero) vector v such that $ABv = \lambda v$. Applying B to the two sides of this equation we see that Bv is an eigenvector of BA corresponding to eigenvalue λ. Thus every eigenvalue of AB is an eigenvalue of BA. This argument gives no information about the (algebraic) multiplicities of the eigenvalues that the earlier five proofs did. However, following the same argument one sees that if $v_1, \quad , v_k$ are linearly independent eigenvectors for AB corresponding to a nonzero eigenvalue λ, then $Bv_1, \quad Bv_k$ are linearly independent eigenvectors of BA corresponding to the eigenvalue λ. Thus a nonzero eigenvalue of AB has the same *geometric multiplicity* as it has as an eigenvalue of BA. This may not be true for a zero eigenvalue. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $BA = O$. Both AB and BA have one eigenvalue zero. Its geomet-

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ric multiplicity is one in the first case and two in the second case.

Proof 7. We want to show that a complex number z is an eigenvalue of AB if and only if it is an eigenvalue of BA. In other words, $(zI - AB)$ is invertible if and only if $(zI - BA)$ is invertible. This is certainly true if $z = 0$. If $z \neq 0$ we can divide A by z. So, we need to show that $(I - AB)$ is invertible if and only if $(I - BA)$ is invertible. Suppose $I - AB$ is invertible and let $X = (I - AB)^{-1}$. Then note that

$$\begin{aligned}(I - BA)(I + BXA) &= I - BA + BXA - BABXA \\ &= I - BA + B(I - AB)XA \\ &= I - BA + BA = I\end{aligned}$$

Thus $(I - BA)$ is invertible and its inverse is $I + BXA$.

This calculation seems mysterious. How did we guess that $I + BXA$ works as the inverse for $I - BA$? Here is a key to the mystery. Suppose a, b are numbers and $|ab| < 1$. Then

$$\begin{aligned}(1 - ab)^{-1} &= 1 + ab + abab + ababab + \\ (1 - ba)^{-1} &= 1 + ba + baba + bababa +\end{aligned}$$

If the first quantity is x, then the second one is $1 + bxa$. This suggests to us what to try in the matrix case.

This proof gives no information about multiplicities of eigenvalues – algebraic or geometric – since it does not involve either the characteristic polynomial or eigenvectors. This apparent weakness turns into a strength when we discuss operators on infinite dimensional spaces.

Let $\mathcal{H}$ be the Hilbert space l_2 consisting of sequences $x = (x_1, x_2, \ .)$ for which $\sum_{j=1}^{\infty} \|x_j\|^2 < \infty$. Let A be a bounded linear operator on $\mathcal{H}$. The *spectrum* of A is the set $\sigma(A)$ consisting of all complex numbers λ such that $(A - \lambda I)^{-1}$ exists and is a bounded linear operator.

The *point spectrum* of A is the set $\sigma_{\text{pt}}(A)$ consisting of all complex numbers λ for which there exists a nonzero vector v such that $Av = \lambda v$. In this case λ is called an eigenvalue of A and v an eigenvector. The set $\sigma(A)$ is a nonempty compact set while the set σ_{pt} can be empty. In other words, A need not have any eigenvalues, and if it does the spectrum may contain points other than the eigenvalues (Unlike in finite-dimensional vector spaces, a one-to-one linear operator need not be onto: and if it is both one-to-one and onto its inverse may not be bounded.)

Unlike in finite-dimensional vector spaces, a one-to-one linear operator need not be onto: and if it is both one-to-one and onto its inverse may not be bounded.

Now let A, B be two bounded linear operators on $\mathcal{H}$. Proof 7 tells us that the sets $\sigma(AB)$ and $\sigma(BA)$ have the same elements with the possible exception of zero. Proof 6 tells us the same thing about $\sigma_{\text{pt}}(AB)$ and $\sigma_{\text{pt}}(BA)$. It also tells us that the geometric multiplicity of each nonzero eigenvalue is the same for AB and BA. (There is no notion of determinant, characteristic polynomial and algebraic multiplicity in this case.)

The point zero can behave differently now. Let A, B be the operators that send the vector $(x_1, x_2, \ \ .)$ to $(0, x_1, x_2, \ \ .)$ and $(x_2, x_3, \ \ .)$ respectively. Then BA is the identity operator while AB is the orthogonal projection onto the space spanned by vectors whose first coordinate is zero. Thus the sets $\sigma(AB)$ and $\sigma_{\text{pt}}(AB)$ consist of two points 0 and 1, while the corresponding sets for BA consist of the single point 1.

A final comment on rectangular matrices A, B. If both products AB and BA make sense, then the nonzero eigenvalues of AB and BA are the same. Which of the proofs shows this most clearly ?