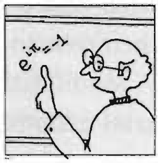

In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Fibonacci Numbers, Seating Arrangements and Stair-climbing

Introduction

The Fibonacci sequence $\{F_n\}$ is defined recursively by $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. For convenience we sometimes write $F_0 = 0$.

We obtain different identities involving the Fibonacci numbers including the well-known Binet formula. It turns out that every positive integer can be expressed as a sum of distinct Fibonacci numbers. Fibonacci numbers have divisibility properties with some interesting consequences. For example, the GCD of two Fibonacci numbers is itself a Fibonacci number. We discuss matrices formed by these GCDs; they turn out to be positive definite. Finally, we consider combinatorial interpretations of a number of identities.

Binet's Formula and Some Identities

Keywords
Fibonacci numbers, GCD-closed matrices, Binet's formula.

To arrive at the so-called Binet formula, we work with the defining linear recurrence relation, exploiting the fact that it has constant coefficients. Write $f(t) = \sum_{n=1}^{\infty} F_n t^n$ and regard the entity f as a purely formal power series (so considerations of convergence need not trouble us). Then

$$\begin{aligned} f(t) &= t + t^2 + \sum_{n=1}^{\infty} (F_n + F_{n+1})\, t^{n+2} \\ &= t + t^2 + t^2 f(t) + t\,(f(t) - t) \ = \ t + (t + t^2) f(t), \end{aligned}$$

giving

$$f(t) = \frac{t}{1 - t - t^2} = \frac{t}{(1 - \alpha t)(1 - \beta t)},$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Expanding by partial fractions we get

$$f(t) = \frac{\alpha}{\alpha - \beta}\left(\frac{t}{1 - \alpha t}\right) - \frac{\beta}{\alpha - \beta}\left(\frac{t}{1 - \beta t}\right)$$

and this yields Binet's formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}.$$

From this formula we quickly get identities such as:

$$\begin{aligned} F_{n+1}F_{n-1} - F_n^2 &= (-1)^n & (1) \\ F_{n+2}F_{n-1} - F_{n+1}F_n &= (-1)^n \text{ (for } n \geq 2), & (2) \\ F_{m+n} &= F_{m-1}F_n + F_m F_{n+1}, & (3) \\ F_{F(n)}^2 + F_{F(n+1)}^2 &= F_{F(n+2)}, F_{F(n-1)} & (4) \\ & \qquad (\text{for } n - 1 \not\equiv 0 \pmod 3)). \end{aligned}$$

(In (4) we have written $F(n)$ for F_n to avoid double subscripts.) For instance consider (1); since $\alpha - \beta = \sqrt{5}$ and $\alpha\beta = -1$ we get

$$F_{n+1}F_{n-1} - F_n^2$$

$$= \frac{1}{5}\left(\alpha^{2n} + \beta^{2n} - \alpha^{n+1}\beta^{n-1}\right.$$

$$\left. -\alpha^{n-1}\beta^{n+1} - \alpha^{2n} - \beta^{2n} + 2\alpha^n\beta^n\right)$$

$$= \frac{1}{5}(\alpha\beta)^n\left(2 - \frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)$$

$$= \frac{1}{5}(-1)^n(2-(-1)3) \;=\; (-1)^n$$

Using (1) we easily get (2) in an inductive manner. The proof of (4) is more challenging. We have

$$F^2_{F(n)} + F^2_{F(n+1)}$$

$$= \tfrac{1}{5}\left(\alpha^{2F(n)} + \beta^{2F(n)} - 2(\alpha\beta)^{F(n)} + \alpha^{2F(n+1)} + \beta^{2F(n+1)} - 2(\alpha\beta)^{F(n+1)}\right)$$

$$= \tfrac{1}{5}\left(\alpha^{2F(n)} + \beta^{2F(n)} + \alpha^{2F(n+1)} + \beta^{2F(n+1)}\right)$$

$$\text{if } n-1 \not\equiv 0 \pmod 3.$$

(We have used the relation $\alpha\beta = -1$ and the fact that $F(n+1)-F(n) = F(n-1)$ is odd if $n-1 \not\equiv 0 \pmod 3$.) The quantity on the right is

$$\frac{1}{5}\left(\alpha^{F(n+2)} - \beta^{F(n+2)}\right)\left(\alpha^{F(n-1)} - \beta^{F(n-1)}\right)$$

$$= \frac{1}{5}\left(\alpha^{F(n+2)+F(n-1)} + \beta^{F(n+2)+F(n-1)} - (\alpha\beta)^{F(n-1)}\left(\alpha^{F(n+2)-F(n-1)} + \beta^{F(n+2)-F(n-1)}\right)\right)$$

$$= \frac{1}{5}\left(\alpha^{2F(n)} + \beta^{2F(n)} + \alpha^{2F(n+1)} + \beta^{2F(n+1)}\right)$$

$$n-1 \not\equiv 0 \pmod 3.$$

(We have used the relations $F(n+2) + F(n-1) = 2F(n+1)$, $F(n+2) - F(n-1) = 2F(n)$.) So (4) is proved.

Here are some more identities which can easily be proved using induction. We leave the proofs as exercises.

$$
\begin{aligned}
F_1 + F_3 + \quad + F_{2n-1} &= F_{2n}, & (5)\\
F_0 + F_1 + F_2 + \quad + F_{n-2} &= F_n, & (6)\\
F_0 + F_2 + F_4 + \quad + F_{2n} &= F_{2n+1}, & (7)\\
F_1^2 + F_2^2 + \quad + F_n^2 &= F_n F_{n+1}. & (8)
\end{aligned}
$$

Remark

Every positive integer can be expressed as a sum of distinct Fibonacci numbers. This is easily shown by subtracting from the given number the largest Fibonacci number not exceeding it, and then proceeding inductively.

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Divisibility Properties

The Fibonacci numbers have some striking divisibility properties. Note firstly that consecutive Fibonacci numbers are coprime. For, if $d \mid F_n$ and $d \mid F_{n-1}$, then the relation that defines the Fibonacci numbers yields $d \mid F_i$ for all $i \leq n$, and this gives $d = 1$.

Theorem 1 $F_m \mid F_n \iff m \mid n$.

Proof. We shall make use of identity (3). Suppose firstly that $m \mid n$, say $n = mk$. We shall show by induction that $F_m \mid F_{mk}$. There is nothing to prove if $k = 1$. If the divisibility holds for some k, then consider the relation

$$F_{m(k+1)} = F_{mk+m} = F_{mk-1}F_m + F_{mk}F_{m+1}.$$

Since F_m is a divisor of both terms on the right, it follows that $F_m \mid F_{mk+m}$.

For the converse, assume that $m > 2$, $F_m \mid F_n$ and $m \neq n$. By (3) we have

$$F_n = F_{n-m+m} = F_{n-m-1}F_m + F_{n-m}F_{m+1}.$$

If $n > 4$ and F_n is prime, then n is prime.

As $F_m \mid F_n$, we get $F_m \mid F_{n-m}F_{m+1}$; but F_m, F_{m+1} are coprime, so we get $F_m \mid F_{n-m}$. Proceeding recursively we get that if $n = qm + r$ with $0 < r < m$, then $F_m \mid F_r$, a contradiction since $F_r < F_m$. Therefore $r = 0$, and we get that $m \mid n$. QED

An obvious implication of the theorem is: if $F_d \mid F_m$ and $F_d \mid F_n$, then $d \mid \gcd(m, n)$.

Corollary 1 *If $n > 4$ and F_n is prime, then n is prime.*

Proof. If $n > 4$ is not prime then it possesses a proper divisor $d > 2$. (Take d to be the largest proper divisor of n.) This yields $2 \leq F_d < F_n$ and $F_d \mid F_n$, contradicting the supposition that F_n is prime. So n must be prime.

Theorem 2 *If* $\gcd(m, n) = k$, *then* $\gcd(F_m, F_n) = F_k$.

Proof. The logic used is similar to that used in the proof of Theorem 1.

Let $d \mid F_m$, $d \mid F_n$. From the relation $F_m = F_{m-n-1}F_n + F_{m-n}F_{n+1}$, which follows from (3), we see that $d \mid F_{m-n}$. (We use the fact that F_n and F_{n+1} are coprime.) Repeated application of this fact shows that if d divides the gcd of F_m, F_n then it divides F_k, where k is the gcd of m, n. So $\gcd(F_m, F_n)$ divides F_k. On the other hand Theorem 1 implies that F_k divides both F_m and F_n. It follows that $\gcd(F_m, F_n)$ is equal to F_k. QED

Matrix of gcds of the Fibonacci Numbers

Let $S = \{x_1, x_2, \quad, x_n\}$ be a set of distinct natural numbers. We say that S is *greatest common divisor closed* (GCDC for short) if $\gcd(x_i, x_j) \in S$ for all i, j, and *divisor closed* (DC for short) if the set of divisors of the numbers in S is S itself. The GCD matrix $G = (g_{ij})$

of S is defined by $g_{ij} = \gcd(x_i, x_j)$. In the theorem below, φ denotes Euler's totient function ($\varphi(x)$ = the number of positive integers less than or equal to x whose gcd with x is 1); θ may be regarded as a generalization of φ. An important result from number theory needed in the proof is the relation

$$\sum_{d|n} \varphi(d) = n \qquad (\text{all } n \in \mathrm{N}). \tag{9}$$

Theorem 3 *Given a GCDC set $S = \{x_1, x_2, \quad , x_n\}$, where $x_1 < x_2 < \quad < x_n$, there exists a real-valued function θ defined on S such that*

(i) $y = \sum_{x|y, x \in S} \theta(x)$, *for all $y \in S$;*
(ii) $\theta(x) \geq \varphi(x)$ *for all $x \in S$.*

Proof. It suffices to define θ as shown below. Let $S(x_j)$ be the set defined by

$$S(x_j) = \{d \in \mathrm{N} \ : \ \text{least multiple of } d \text{ in } S \text{ is } x_j\}$$

Observe that $x_j \in S(x_j)$ (so each $S(x_j)$ is non-empty) and that the $S(x_j)$ are disjoint. Now define θ by

$$\theta(x_j) = \sum_{d \in S(x_j)} \varphi(d).$$

Note that property (ii) of the theorem is trivially met; we need only to prove (i), which states that $\sum_{x_i|x_r} \theta(x_i) = x_r$ for $1 \leq r \leq n$. This identity clearly holds for $r = 1$ (because x_1 is the least element of S, so $S(x_1)$ consists simply of the divisors of x_1). For $r > 1$, observe that the disjoint union

$$\bigcup_{x_i|x_r} S(x_i)$$

yields the set of divisors of x_r. This follows from the way in which the sets $S(x_i)$ are defined (the fact that S

is GCDC is needed). Now (4.1) is invoked to yield the desired identity. QED

We give an application of these ideas to the GCD matrix of a GCDC set. The function θ is defined as above, and G_n is the GCD matrix of $S = S_n$.

Theorem 4 *For a GCDC set* $S_n = \{x_1, x_2, \quad , x_n\}$, *where the* x_i *are natural numbers with* $x_1 < x_2 < \quad < x_n$, *we have*

$$\det(G_n) = \theta(x_1)\theta(x_2) \quad \theta(x_n).$$

Moreover, G_n *is positive definite.*

Proof. Recall that $g_{ij} = \gcd(x_i, x_j)$. Consider the $n \times n$ matrix A defined by

$$a_{ij} = \begin{cases} \sqrt{\theta(x_i)} & \text{if } x_i \mid x_j, \\ 0 & \text{else.} \end{cases}$$

Let ${}^tA$ denote the transpose of A; then the $(i,j)^{\text{th}}$-element of ${}^tAA$ is

$$\sum_k a_{ki}a_{kj} = \sum_{x_k \mid \gcd(x_i,x_j)} \gcd(x_i, x_j).$$

It follows that $G_n = {}^tAA$, and therefore that $\det(G_n) = \det(A)^2$.

Since A is upper-triangular, its determinant is the product of its diagonal entries, i.e.,

$$\det(A) = \sqrt{\theta(x_1)\theta(x_2) \quad \theta(x_n)}.$$

The stated result follows.

To show that G_n is positive definite note that the principal minors of G_n are just $G(x_r)$ for $r \leq n$. QED

Remark

For another proof that G_n is positive definite, observe that the GCD matrix of a GCDC set S_n is contained as a $n \times n$ submatrix of the $x_n \times x_n$ matrix whose entries are $\gcd(i,j)$ for $i, j \le x_n$. The set $\{1, 2, 3, \quad, x_n\}$ is factor-closed, so its GCD matrix H is positive definite. (This is a known result; see [1] and [2], Chapter 8.) The GCD matrix of S_n is obtained by removing some rows and the corresponding columns from H and is therefore positive definite.

It is not clear whether any simple geometric interpretation can be found to the fact that the GCD matrix of the set $\{F_2, F_3, \quad, F_n\}$ is positive definite.

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Seating Arrangements

Consider the following problem. *A classroom has* 2 *rows of* n *desks per row. The teacher asks each pupil to move to one of the desks immediately to her front, back, left or right. (Not all options will be open to all the pupils.) Find the number of ways in which these transfers can be carried out.*

Let g_n be the required number. We shall solve the problem by finding a recursive relationship for g_n. Empirically we find that $g_1 = 1$, $g_2 = 4$ and $g_3 = 9$. Is g_n equal to n^2? Let us not be hasty!

We count g_n as follows.

- The two pupils in the first column may interchange places, and then the remaining pupils can change places in g_{n-1} ways.
- The four pupils in the first two columns can interchange places amongst themselves in three additional ways, and the remaining pupils can then interchange places in g_{n-2} ways.
- For $i = 3, 4, \quad n$ there are precisely two ways

If a flight of n stairs is to be climbed, and on each step I climb either 1 or 2 stairs, then the number of different ways of reaching the top is F_{n+1}.

for the pupils in the first i columns to interchange places if there is to be no subset of the columns whose pupils interchange places only amongst themselves; namely, via a clockwise transfer or an anti-clockwise transfer.

The above listing of possibilities easily yields the following relation:

$$g_n = g_{n-1} + 3g_{n-2} + 2\left(g_{n-3} + g_{n-4} + \quad + g_1\right) + 2. \quad (10)$$

This yields $g_4 = 25 = 5^2$, thereby disproving the suspicion that $g_n = n^2$. But we notice that $g_4 = F_5^2$. After some computing we get $g_5 = 64 = 8^2 = F_6^2$. Now we are able to guess that $g_n = F_{n+1}^2$, and once this relation is spotted an inductive proof is quickly obtained.

Stair-climbing

Identities such as (3) and the following:

$$\begin{aligned} F_{2k+2} &= F_{k+1}^2 + 2F_kF_{k+1}, && (11)\\ F_{3k+1} &= F_{k+1}^3 + 2F_k^2F_{k+1} + F_k^2F_{k-1}, && (12)\\ F_{4k+1} &= F_{k+1}^4 + F_k^4 + 2F_k^2F_{k+1}^2 + && (13)\\ & \qquad F_k^2\left(F_{k+1} + F_{k-1}\right)^2 \end{aligned}$$

may also be proved using a combinatorial interpretation to the Fibonacci numbers: *If a flight of n stairs is to be climbed, and on each step I climb either* 1 *or* 2 *stairs, then the number of different ways of reaching the top is* F_{n+1}. This may be shown quite easily, and we leave the proof to the reader. (*Example.* A flight of 3 stairs can be climbed as 111 or 12 or 21, i.e., in $3 = F_4$ different ways.)

Let us see how this interpretation may be used to prove (3), which may be stated alternatively as

$$F_{m+n+1} = F_mF_n + F_{m+1}F_{n+1}.$$

The left side represents the number of ways of climbing a flight of $m + n$ stairs. Consider the different ways of reaching the top and categorize them as type I/type II, according to whether we step upon stair $\#m$ or not. Now count both categories. For I: we can reach stair $\#m$ in F_{m+1} ways and climb from stair $\#m$ to stair $\#(m+n)$ in F_{n+1} ways; so category I has $F_{m+1}F_{n+1}$ paths. In II, we must necessarily step upon stair $\#(m-1)$. We can reach this stair in F_m different ways; on reaching this, we take a double step to stair $\#(m+1)$, and then climb from stair $\#(m+1)$ till stair $\#(m+n)$ in F_n different ways. Therefore there are F_mF_n different paths in category II. It follows that $F_{m+n+1} = F_mF_n + F_{m+1}F_{n+1}$, as required.

Identity (11) may be proved as follows. We consider climbing a flight of $2k + 1$ stairs (F_{2k+2} different possibilities). We may or may not step upon stair $\#(k+1)$. If we do, then we have $F_{k+2}F_{k+1}$ different paths available. If we do not, then we must reach stair $\#k$ (in F_{k+1} possible ways), take a double step to stair $\#(k+2)$, and then climb the remaining $(k-1)$ stairs (in F_k different ways). It follows that

$$F_{2k+2} = F_{k+2}F_{k+1} + F_{k+1}F_k = F_{k+1}^2 + 2F_kF_{k+1},$$

after substituting for F_{k+2}.

Similar schemes may be found for (12) and (13); these we leave to the reader.

Suggested Reading

[1] B V R Bhat, On Greatest Common Divisor Matrices and their Applications, *Linear Algebra and its Applications*, Vol. 158, 1991.

[2] G Polya and G Szego, *Problems in Analysis*, Volumes 1-2, Springer-Verlag, 1971.

[3] R Honsberger, *In Polya's Footsteps* (Math Association of America).

[4] I Niven and S Zuckermann, *An Introduction to the Theory of Numbers*, John Wiley, 1960.