# Newton's Method

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If one follows in the footsteps of Newton, 'almost all' roads lead to Rome (or is it London?).

## Zeroing in

If one wants to solve the nonlinear equation $f(x) = 0$ for a 'nice' (i.e., continuously differentiable) function $f : R^d \to R^d$, an old favourite is the Newton (or Newton–Raphson) method.[1] Letting $Df(x) = [[\frac{\partial f_i}{\partial x_j}(x)]]$ denote the Jacobian matrix of $f$, the method can be described as follows: If one moved from $x_0 \in R^d$ to a 'nearby' $x_1 \in R^d$, where $f(x_1) = 0$, then $f(x_1)$ approximately equals $f(x_0) + Df(x_0)(x_1 - x_0)$. Then $x_1$ approximately equals $x_0 - Df(x_0)^{-1}f(x_0)$ (assuming, of course, that the inverse exists). This suggests the iteration

$$x_{n+1} = x_n - Df(x_n)^{-1}f(x_n),$$

which is the Newton–Raphson scheme. What one is really doing at step $n$ is to solve *exactly* the 'linearized' equation $f_n(x) = 0$ to find $x_{n+1}$, where $f_n$ is the linear approximation to $f$ at $x_n$ given by $f_n(x) = f(x_n) + Df(x_n)(x - x_n)$. *Figure* 1 shows what happens in a one dimensional example.

[1] Raphson, an eminent scientist of his day and a friend of Newton, published the method in his book *Analysis Aequationum Universalis* in 1690. Newton described it in *Method of Fluxions* written in 1671, but published in 1736 [1].

But is this example representative? Consider for example the situation in *Figure* 2, where the iterations will keep oscillating. Thus convergence to a zero of $f$ is certainly not guaranteed in general. But then, is *Figure* 2 typical? One can't help suspecting that it is 'tailored' to confuse the algorithm, so perhaps for most $f$ and most initial conditions it will still work.

Keywords
Implicit function theorem, Sard's theorem, regular value, Newton–Raphson scheme.

Our aim here is to present one result that confirms this intuition, albeit for the continuous time counterpart of

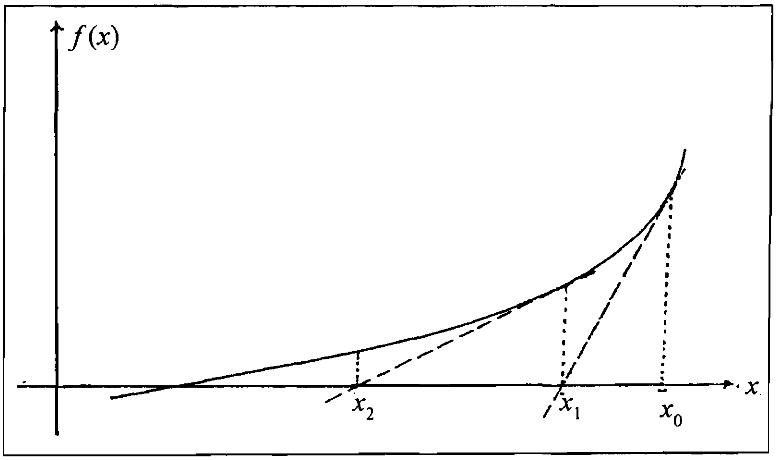


Figure 1.

the above scheme. To arrive at this, consider the variation

$$x_{n+1} = x_n - aDf(x_n)^{-1}f(x_n), \qquad (1)$$

for a small $a > 0$. This 'incremental' version is already an improvement over the original scheme in most situations because it buys a more graceful behaviour of the algorithm at the expense of its speed. Dividing the above equation through by $a$ and letting $a \downarrow 0$, we get

$$\dot{x}(t) = h(x(t)) \triangleq -Df(x(t))^{-1}f(x(t)). \qquad (2)$$

This is the continuous time Newton scheme we shall work with. Alternatively, one may view (1) as a dis-

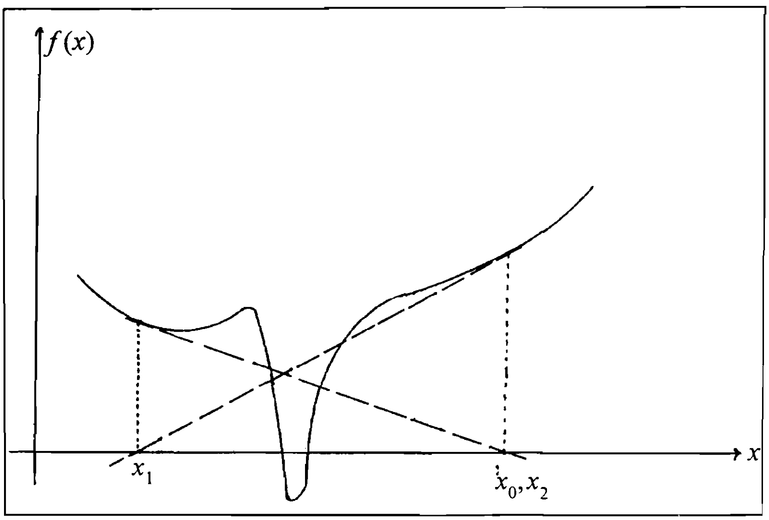


Figure 2.

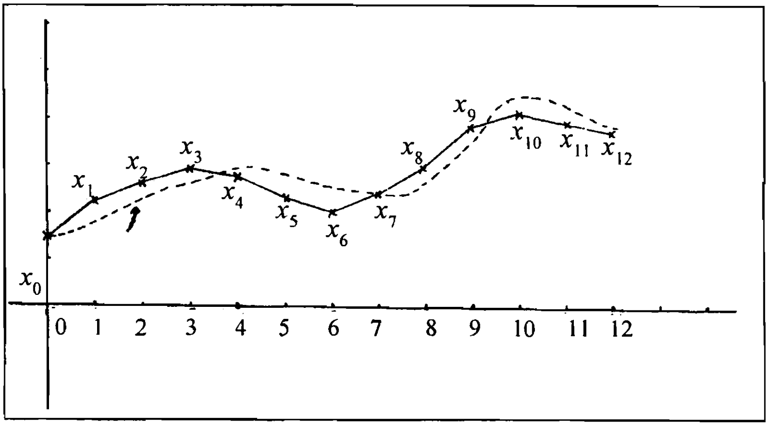


*Figure 3.*

cretization or 'Euler scheme' for approximate solution of the o.d.e. (2) on a computer. (See *Figure* 3 for a schematic where the solid line is a piecewise linear interpolation of (1) and the broken line is a trajectory of (2).)

## Towing the Line

Let $E = \{x : f(x) = 0\}$ be bounded. We shall consider a bounded domain $D$ containing $E$ in its interior and with a smooth boundary $\partial D$ satisfying: for all $x \in \partial D$, $Df(x)$ is full rank, and $h(x)$ is pointing inwards (see *Figure* 4). Set $\bar{D} = D \bigcup \partial D$.

Let $S = \{x : ||x|| = 1\}$ denote the unit sphere and define $g : D - E \to S$ by $g(x) = \frac{f(x)}{||f(x)||}$. Let $y = f(x_0) \in S$ for some $x_0 \in \partial D$ be a point such that for any $x$ with $y = g(x)$, $Dg(x)$ is full rank. (Such a $y$ is said to be *regular*.) Consider the set $C = \{x \in \bar{D} - E : g(x) = y\}$. For any point $p$ in C, since $Dg(p)$ is full rank, $g(x) = y$ is a set of $(d - 1)$ independent constraints in a neighbourhood of $p$. (Recall that $S$ is a $(d - 1)$–dimensional object.) Thus one expects $C$ to be a one dimensional curve in a neighbourhood of $p$, extending on either side of $p$. That this is indeed so is guaranteed by the celebrated *implicit function theorem* (IFT) of advanced calculus. We can then patch up these one dimensional pieces and claim that $C$ is a union of one dimensional curves. We shall

IFT applied at $p$ guarantees us that the patch of $C$ around $p$ is a decent stand alone one dimensional curve.

focus on the one that starts at $x_0$. Once again, the IFT allows us to eliminate some potential pathologies. For example, this curve cannot intersect itself again (or come arbitrarily close to itself) at some point $p$ on it, because IFT applied at $p$ guarantees us that the patch of $C$ around $p$ is a decent stand alone one dimensional curve. Similarly, the curve cannot end abruptly at some point $p$ in the interior of $\bar{D} - E$, because IFT applied at $p$ tells us that we can extend it a bit further. Thus this curve can either end up in $E$ or back in $\partial D$.

We shall now show that this curve is in fact a trajectory of the continuous time Newton scheme described above and will indeed end up in $E$, as one hopes it would. The proof of the first claim is an exercise in calculus. Parametrize the curve as $x(t), t \geq 0$, with $x(0) = x_0$. Then $g(x(t)) = y$. Differentiating w.r.t. $t$,

$$\frac{d}{dt} g(x(t)) = \frac{d}{dt}\left(\frac{f(x(t))}{||f(x(t))||}\right) = 0.$$

The reader is invited to work this through and show that it reduces to

$$(I - yy^T)Df(x(t))\dot{x}(t) = 0,$$

where $I$ is the identity matrix. Thus by elementary linear algebra, $Df(x(t))\dot{x}(t) = \beta(t)y$ for some scalar $\beta(t)$. Setting $\alpha(t) = \frac{\beta(t)}{||f(x(t))||}$, we have $Df(x(t))\dot{x}(t) = \alpha(t)f(x(t))$. Since $Df(x(t))$ is nonsingular, $\alpha(t) \neq 0$ and thus does not change sign as $t$ varies. Also,

$$\dot{x}(t) = \alpha(t)Df(x(t))^{-1}f(x(t)) = -\alpha(t)h(x(t)).$$

Since $h(x_0)$ is directed inwards at $x_0$, $\alpha(0) < 0$, hence $\alpha(t) < 0$ for all $t$. Letting $\gamma(t) = -\alpha(t)$, we conclude that $x(\cdot)$ is in fact a trajectory of the differential equation

$$\dot{x}(t) = \gamma(t)h(x(t)).$$

But then it is also a trajectory of the continuous Newton scheme

$$\dot{x}(t) = h(x(t)).$$

This is because putting a positive scalar valued function $\gamma(\cdot)$ up front on the right hand side of a differential equation does not change its trajectories, it only changes the *speed* with which it moves along the trajectories.

Having established that we are indeed moving down a trajectory of the algorithm, it remains to show that it ends up in $E$. If not, our earlier remarks imply that it must end up back in $\partial D$. But following the orientation of $h(x(t))$ along $x(t)$ beginning with '*inwards*' at $x_0$, we find that it must be '*outwards*' at the point where it hits $\partial D$ again, a contradiction. (See *Figure* 4.) Thus $x(t)$ must converge to $E$, the desired solution set.

By Sard's theorem, the set of points which are not regular has zero volume in $S$.

## Measure for Measure

Can we do this for all possible $x_0$? Well, we almost can, as we argue next. Note that the only serious restriction we slipped in was that $y$ be regular. By a well-known theorem due to Sard, such points have full measure in

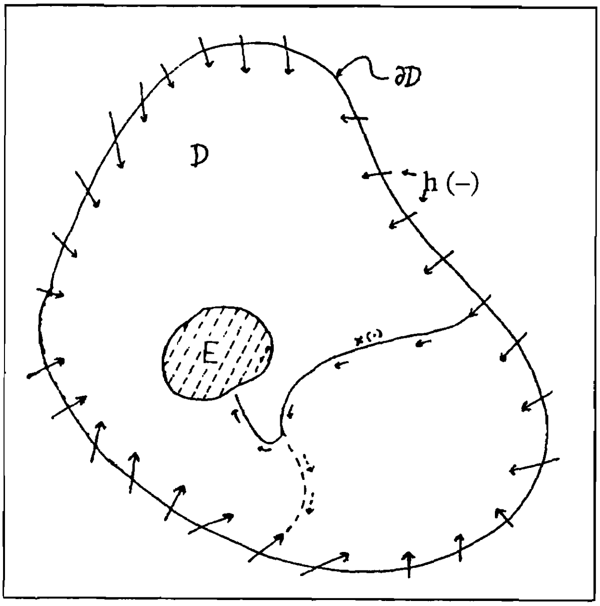


*Figure 4.*

$S$. In other words, the set $\Lambda$ of points which are *not* regular (i.e., are *singular*) has zero 'volume' in $S$. (For those unfamiliar with measure theory, a point (resp., a line, a plane) has zero volume in a line (resp., a plane, the three dimensional space).)

Now let $g_0 : \partial D \to S$ denote the restriction of $g$ to $\partial D$. From our assumption that $Df$ be nonsingular at points of $\partial D$ and the fact that $h = -(Df)^{-1}f$ is transversal (i.e., *not* tangential) to $\partial D$, it follows that $g_0$ is 'nonsingular' in a neighbourhood of any point of $\partial D$ in the sense that it will map a set of positive $(d-1)$–dimensional volume in $\partial D$ to a set of positive $(d-1)$–dimensional volume in $S$. Thus the set of points in $\partial D$ that get mapped into $\Lambda$ must have zero volume in $\partial D$. Thus for 'almost all' $x_0$ in $\partial D$, the algorithm takes us to $E$.

I have avoided the ticklish issue of defining the 'volume' in $S$ or $\partial D$. Suffice to say that for 'nice' manifolds, there will be a natural recipe to do this (e.g., think of the surface area on a three dimensional sphere). Also, there is considerable flexibility permitted here because of the fact that we don't care what numerical values our definition of volume assigns to sets of positive volume as long as it is clear what the sets of zero volume ('measure') are.

To conclude, what I have attempted here is a bowdlerized version of [2]. The word '*bowdlerize*' comes from Thomas Bowdler, an American gentleman who came up with a version of Shakespeare in which he eliminated all four letter words[2] so as to render it suitable for reading in 'decent households'! I have likewise eliminated fourteen letter words like *diffeomorphism*, but in essence it is the same treatment as that of [2].

## Suggested Reading

[1] D Bertsekas, *Nonlinear Programming* (2nd edition), Athena Scientific, Belmont, Mass, 1999.

[2] S Smale, A convergent process of price adjustment and global Newton methods, *Journal of Mathematical Economics*, Vol.3, pp.107-120, 1976.

[2] Figuratively speaking, of course. Apparently the average word length in Shakespeare is indeed close to four !

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