In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Some Aspects of the Hypotenuse Theorem

This article briefly describes some methods of proof adopted for the famous theorem attributed to Pythagoras.

The proof of the theorem is attributed to Pythagoras by various writers like Proclus (460 BC), Plutarch (first century), Cicero (50 BC), Deogenes Laertius (second century) and Athenaeus (c.300).

Geometric algebra existed in ancient India long before the classical period in Greece, and the Greeks themselves supposed that they had received it from the more ancient East. The *Sulva Sutras* contain many geometrical and trigonometrical results. The *Baudhayana Sulva Sutra* gives an approximate solution for squaring the circle. The *Sulva Sutra* of *Apasthamba* gives rules for the construction of right angles. The hypotenuse theorem (better known now as Pythagoras's theorem) is dealt within the *Taittriya Samhita* under two aspects. In the first aspect the theorem is used to construct the side of a square equal in area to the sum or difference of two given squares. In the second aspect the diagonal of a rectangle is computed. The proof of the hypotenuse theorem is attributed to Pythagoras by various writers [1] such as Proclus (460 BC), Plutarch (1st century), Cicero (50 BC), Deogenes Laertius (2nd century) and Athenaeus (300 AD). Apart from this there is no other strong historical evidence to believe that Pythagoras was the first to prove the proposition. But the *Sulva Sutras* contain explicit statements of both aspects of the proposition with indications of proofs, and it would appear that geometric algebra of Greece came from ancient India. Efforts are being made today to rectify the prevailing theories. (see [2], [3], [4]).

According to [5], only seven *Sulva Sultras* are known today: the *Baudhayana, Apasthamba, Katyayana, Manava, Maitrayana, Varaha* and the *Vadhula*, and of these the first three have mathematical significance. They explain simple geometrical constructions of squares, rectangles, parallelograms and trapezia. The date of the oldest *Sulva Sutra*, the *Baudhayana*, has been given explicitly as 800 BC. Pythagoras lived from 572 BC till 501 BC and so the *Sulvas* come prior to him. It is interesting that the construction of a right angled triangle with all sides rational was given much later by *Brahmagupta* (7th century AD); the general solution given by him is $m^2 - n^2, 2mn, m^2 + n^2$, where m and n are rational numbers with $m > n$. Pythagoras gives the solution $2n + 1, 2n^2 + 2n, 2n^2 + 2n + 1$, but this is not a general solution. The *Katyayana* gives a method for construction of a square whose area is n times that of a given square. It is important to note that the geometry contained in the *Sulvas* is not based on a system of axioms and postulates like that of Euclid's geometry. The hypotenuse theorem appears in the *Sulvas* in the following form.

The diagonal of a rectangle gives an area equal to the sum of the areas given by its length and its breadth.

There are numerous proofs for this proposition and they are known under different names. Many different configurations are made use of in these proofs.

It would appear that Pythagoras was not the first to

Suggested Reading

[1] D E Smith, *History of Mathematics*, Volumes 1 & 2, Dover Publications, 1953.

[2] A Seidenberg, The ritual origin of geometry, *Archive for History of Exact Sciences*, Vol. 1, 488-527, 1962.

[3] A Seidenberg, Did Euclid's Elements, Book I, develop geometry axiomatically?, *Archive for History of Exact Sciences*, Vol. 14, 1975.

[4] A Seidenberg, The Origin of Mathematics, *Archive for History of Exact Sciences*, Vol. 18, 301-342, 1978.

[5] C N Srinivasiyengar, *A History of Ancient Indian Mathematics*, World Press Private Limited, 1967.

[6] H A Freebury, *A History of Mathematics*, Macmillan, 1958.

[7] Sri Bharatikrsna Tirthaji Maharaja, *Vedic Mathematics*, Motilal Banarsidas Publishers, 1995.

[8] Charles Singer, *A Short History of Scientific Ideas*, Oxford University Press, 1959.

Baudhayana had derived Hypotenuse theorem and calculated the value of π much before Pythagoras.

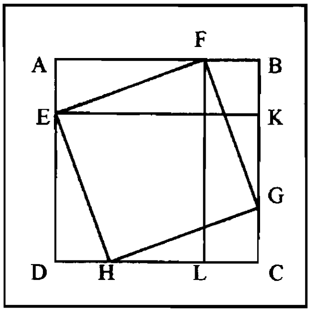


Figure 1.

prove the proposition. Algebraic identities are treated geometrically in the *Sulvas*, and so also are various geometrical constructions which indicate a deep knowledge of properties of similar figures (see *Box* 1). *Baudhayana* derived the hypotenuse theorem and also estimated the value of π much in advance to Pythagoras.

In *Figure* 1, $ABCD$ is a square, and E, F, G, H are points on its sides so that $BF = BK = CG = CL = DH = AE$. Then, we have:

$$\begin{aligned}\text{Sq.}ABCD &= \text{Sq.on}DL + \text{Sq.on}BF + 4 \;\; \text{Area } \triangle AEF \\ &= AF^2 + AE^2 + 4(\triangle AEF), \\ \text{Sq. } ABCD &= EF^2 + 4(\triangle AEF),\end{aligned}$$

therefore $AF^2 + AE^2 = EF^2$. The prevailing view is that the proposition must have been proved in some such manner by *Baudhayana*, because similar geometric constructions are found in the *Sulvas*.

Euclid's Proof. The following proof is believed to be Euclid's (*Figure* 2). Draw $CR \perp BA$; draw $AM, BN \perp AB$ and equal to AB; draw $AL \perp AC$ and equal to AC; draw $BK \perp BC$ and equal to BC. Let $BC = a, CA = b, AB = c, AR = x, BR = y$; then $c = x + y$. Triangles

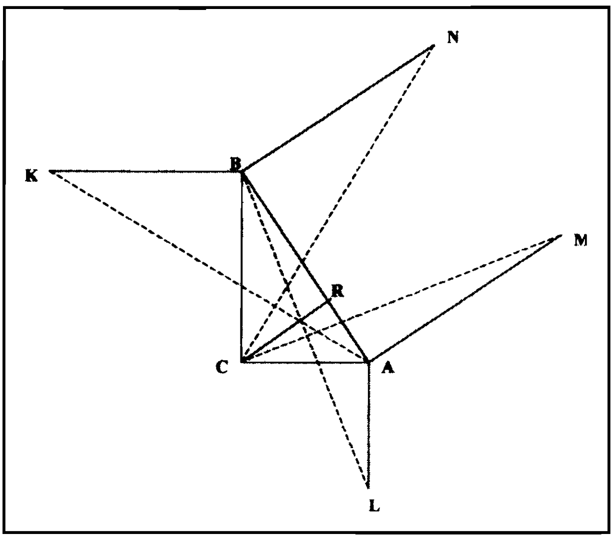
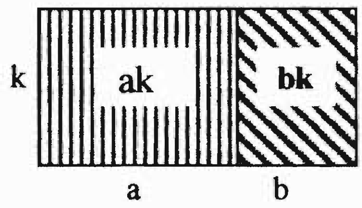


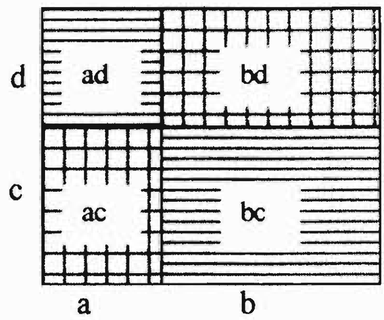
Figure 2.

Box 1. Geometrical Algebra.

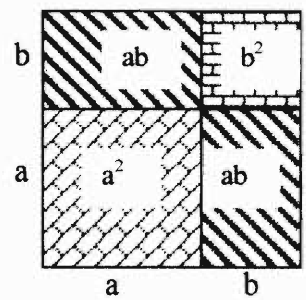
$$(a+b)k = ak + bk$$



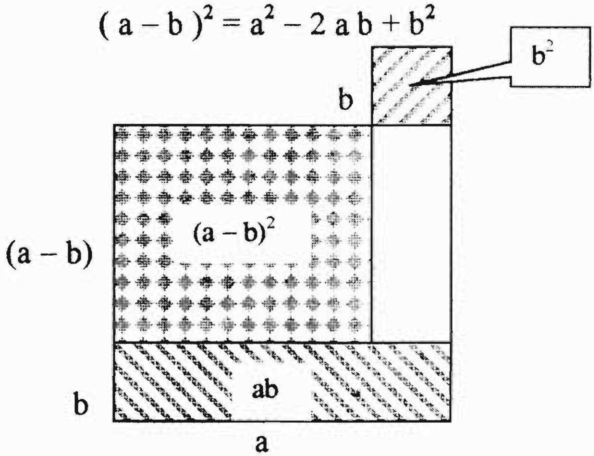
$$(a+b)(c+d) = ac + bc + ad + bd$$



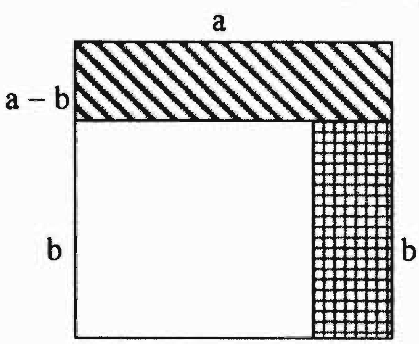
$$(a+b)^2 = a^2 + 2ab + b^2$$

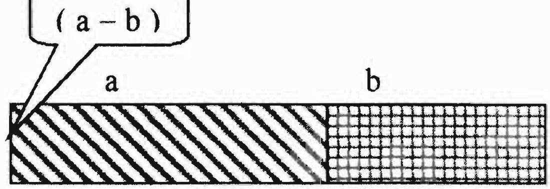


$$(a-b)^2 = a^2 - 2ab + b^2$$



$$a^2 - b^2 = (a+b)(a-b)$$





Box 2. Terms Defined in Euclidean Geometry

Point: A point is that which has no part.
Line: Length without breadth.
Straight line: A line which is evenly with the points on itself.
Plane surface: A surface which lies evenly with the straight lines on itself.
Angle: A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line. When the lines containing the angle are straight, the angle is called rectilineal.
Circle: A plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another; the point is called the centre of the circle.
Parallel lines: Straight lines, which being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

LAB and CAM are congruent, and their areas are $b^2/2$ and $cx/2$, respectively; so $b^2 = cx$. In the same way we get $a^2 = cy$, from the congruence of triangles KBA and NBC. Now addition yields $a^2 + b^2 = c^2$.

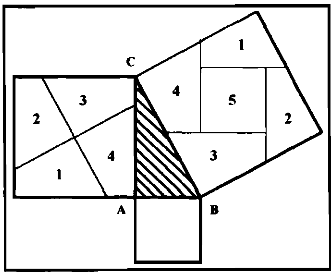
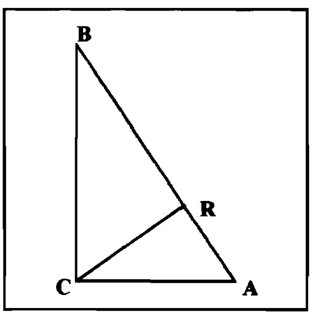


Figure 3.

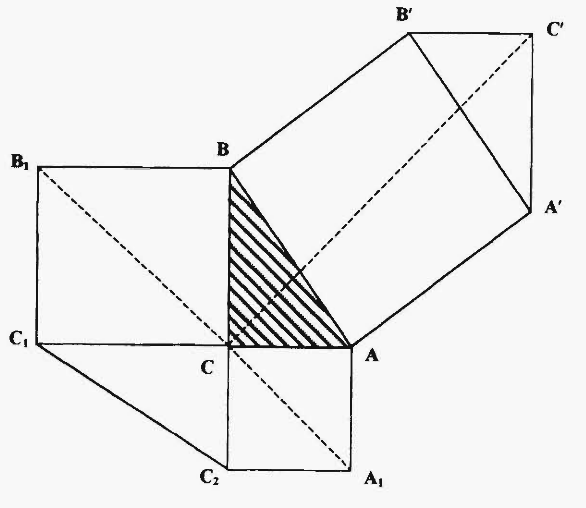
Euclid (ca. 330 BC-260 BC) was a Greek mathematician. He is called the father of geometry, yet little is known about him. His treatise, *The Elements*, comprising thirteen volumes, is still a basis for our school geometry texts. Its material is arranged methodically and logically, with definitions, axioms and postulates. It is interesting to read some of these definitions (see ([1]: *Box* 2). It is difficult to assess whether these texts are Euclid's own or compilations.

Perigal's dissection On a thin cardboard (*Figure* 3) draw a right-angled triangle ABC and squares on the three sides. Choose one of the smaller squares and make cuts as shown (they are perpendicular to one another). The four pieces together with the other small square exactly make up the square on the hypotenuse.

Figure 4.



Similarity of Triangles If two triangles are equiangular then their corresponding sides are proportional. In *Figure* 4, triangles ABC and CBR are similar, as also ABC and ACR. This yields $a^2 + b^2 = c^2$.



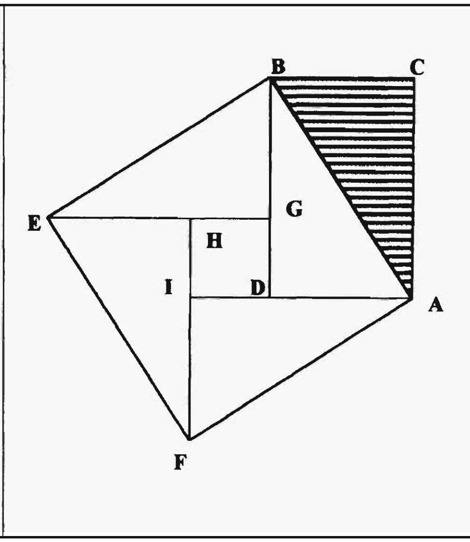


Figure 5. (left)
Figure 6. (right)

Leonardo da Vinci's Proof. In *Figure* 5, $BB'CC'$ and BAA_1B_1 are congruent, as are $B_1C_1C_2A_1$ and $C'A'AC$. A comparison of areas makes it evident that $a^2 + b^2 = c^2$ ([6]); see *Box* 3.

Vedic Proof. Reference [7] claims that there are many 'Vedic proofs' of the hypotenuse theorem, simpler than those given by Euclid. The proof given by Bhaskaracharya (*Box* 4) is based on congruence of triangles (*Figure* 6). Triangles ABD, AIF, EHF and BEG are congruent to one another, and the areas of these four triangles together with that of square $GHID$ make the area of square $ABEF$ Therefore $c^2 = 4(ab/2) + (b - a)^2 = a^2 + b^2$.

A related proof is contained in *Figures* 7, 8. Triangles marked 1, 2, 3, 4, 5, 6, 7, 8 are congruent right angled triangles, and it is immediate that $A = B + C$.

Other Material in the 'Sulva Sutras'. It is appropriate to note here that the *Sulva Sutras* contain many other results besides the hypotenuse theorem. We mention a few of them. The (approximate) squaring of a circle is accomplished in these texts, and the reverse

Box 3. Leonardo da Vinci

An Italian artist, scientist and mathematician of great eminence, famous as a painter, sculptor, goldsmith and architect. He published an edition of Euclid in 1509. In physics he worked on finding the centre of gravity of a pyramid, theory of inclined planes, friction, etc.

Box 4. Bhaskaracharya (12th century)

Along with Aryabhatta (4th century) and Brahmagupta (7th century), he is the most well-known mathematician of ancient India. He wrote a famous four-part text *Siddhanta Siromani* comprising the *Leelavati, Bijaganitam, Goladhyaya* and *Grahaganita*. He anticipated some of the ideas of modern differential calculus 500 years before Newton. He also worked on the indeterminate equation $Nx^2 + 1 = y^2$ where N is a positive integer. This equation known as Pell's equation due to a historical error was studied much later by Lagrange. Five centuries earlier, Brahmagupta too had worked on this equation; see [4].

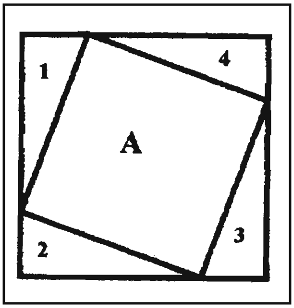


Figure 7.

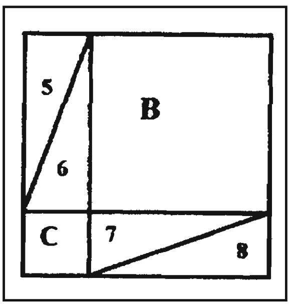


Figure 8.

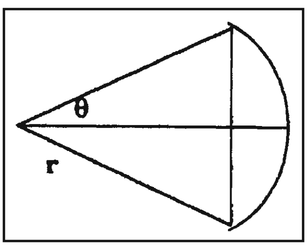


Figure 9.

problem too is solved; see [5]. The approximation used in this construction is

$$\pi \approx \left(\frac{6}{2+\sqrt{2}}\right)^2$$

or $\pi \approx 3.08831$. *Apasthamba* observes that the circle thus constructed is '**anitya**' or 'approximate'. This suggests that the notion of irrational number may have been known in ancient India. In this connection the value given by *Aryabhatta* deserves mention: 'add 4 to 100, multiply by 8 and add 62000, this being the approximate value of the circumference of a circle with a diameter of 20000'; this yields

$$\pi \approx \frac{62832}{20000} = 3.1416.$$

In the reverse problem, with d equal to the diameter of the given circle, the side a of the square is given by

$$2a = \frac{7}{8}d + \left[\frac{d}{8} - \left\{\frac{28}{8\times 20}d + \frac{d}{8\times 29}\left(\frac{1}{6} - \frac{1}{6\times 8}\right)\right\}\right].$$

This also yields an estimate for $\sqrt{2}$; for details refer to [5].

The *Sulvas* also give ways for computing the sines of angles (called *jya* in ancient India, from the Sanskrit word for 'chord').