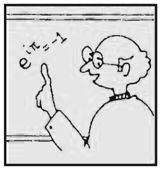
Classroom



In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Logarithm and agM

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In [1] we had discussed the evaluation

$$\int_0^{\pi/2}(a^2\sin^2\theta + b^2\cos^2\theta)^{-1/2}d\theta = \frac{\pi}{2M(a,b)}.$$

Here $M(a,b)$ denotes Gauss's arithmetico-geometric mean of a and b. Recall that $M(a,b)$ is defined to be the common limit approached by the two sequences of arithmetic and geometric means defined starting from 'a' and 'b'.This evaluation was accomplished by a change of variables known as Gauss's transformation which allows one to pass from the afore-mentioned integral to a similar one involving the arithmetic and geometric means of a, b. In fact, a similar invariance property works (as noticed first by Carlson) for the integral $L(a,b) = \int_0^{\pi/2}\log(a\sin^2\theta + b\cos^2\theta)d\theta$. Observe that this integral represents an average of the logarithm over the interval (a,b). We take $a, b \geq 0, a+b > 0$. Take $\theta = \phi/2$ and break up the integral into two integrals as

$$L(a,b) = \frac{1}{2}\int_0^{\pi}\log(\frac{a+b}{2} - \frac{a-b}{2}\cos\phi)d\phi$$

$$= \frac{1}{2}\int_0^{\pi/2} \log(\frac{a+b}{2} - \frac{a-b}{2}\cos\phi)d\phi$$
$$+ \frac{1}{2}\int_{\pi/2}^{\pi} \log(\frac{a+b}{2} - \frac{a-b}{2}\cos\phi)d\phi.$$

Putting $\pi - \phi$ in place of ϕ in the second integral and adding, we get

$$L(a,b) = \frac{1}{2}\int_0^{\pi/2} \log((\frac{a+b}{2})^2 - (\frac{a-b}{2})^2\cos^2\phi)d\phi$$
$$= \frac{1}{2}L((\frac{a+b}{2})^2, ab).$$

Once again, as in [1], this will be used repeatedly.

Starting with $a_0 = a, b_0 = b$, put $a_{n+1} = (\frac{a_n+b_n}{2})^2, b_{n+1} = a_n b_n$. So, $L(a,b) = 2^{-n}L(a_n,b_n) = 2^{-n}\log a_n + 2^{-n} L(1, b_n/a_n)$. for all n.

To find out the limit, we take $c_n = \frac{1}{2}(\sqrt{a_n} + \sqrt{b_n})$, $d_n = \frac{1}{2}(\sqrt{a_n} - \sqrt{b_n})$ and $e_n = d_n/c_n$. Then, $c_{n+1} = c_n^2, d_{n+1} = d_n^2, e_{n+1} = e_n^2$. Thus, we get (since $e_1 = d_0^2/c_0^2 < 1$) as $n \to \infty$, that

$$\frac{b_n}{a_n} = (\frac{1-e_n}{1+e_n})^2 \to 1 \quad \frac{c_n^2}{a_n} \to 1\ ,\ e_n \to 0.$$

For $a, b > 0$, the integrand in the definition of $L(a,b)$ is jointly continuous in a, b, θ and $L(a,b)$ is continuous in a and b. As $L(1,1) = 0$, we obtain

$$L(a,b) = \lim_{n\to\infty} 2^{-n}\log a_n = 2\ \lim_{n\to\infty} 2^{-n}\log c_n.$$

As $2^{-n}\log c_n$ is independent of n by the above remark that $C_{n+1} = C_n^2$ it follows that

$$L(a,b) = 2\ \log\frac{\sqrt{a}+\sqrt{b}}{2}.$$

The argument when one of a, b is zero requires only a minor modification.

[1] B Sury, *Resonance*, Vol.5, No.8, p.74, 2000.