On Transcendental Functions

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Functions like $\log_e x, e^x, e^z$ and $\log_e z$ are defined and studied systematically. It is observed that e^z is the only complex analytic extension of the real analytic function e^x. It is also observed that the power (analyticity) of e^z is by choice and the beauty (its derivative is itself) of e^z is unexpected.

1. Introduction

Functions like $\log_e x, e^x, x$ real, and e^z $\log_e z, z$ complex are very important in mathematics and physics. The results in this article are not new, but they are presented in a very natural way. It is observed that e^z is the one and only one complex analytic extension of the real analytic function e^x with the preservation of homomorphism.

We use the most important theorems in basic calculus namely, intermediate value theorem, chain rule, Cauchy's theorem in Riemann integration, fundamental theorem of calculus, Taylor series formula, inverse mapping theorem and open mapping theorem for metric spaces to define $\log_e x$ and e^x. In other words, we can say that all these theorems have been proved only after studying $\log e^x$ and e^x.

In Section 2, we study the functions $\log_e x$ and e^x. In Section 3 we study the functions e^z and $\log_e z$. We denote by R the set of real numbers, R_+ the set of positive real numbers, R^* the set of non-zero real numbers, $\mathbb{C}$ the set of complex numbers, $\mathbb{C}^*$ the set of non-zero complex numbers, N the set of natural numbers and $\mathbb{Z}$ the set of integers. We know that R and $\mathbb{C}$ are groups with respect to addition and R_+, R^* and $\mathbb{C}^*$ are groups with respect to multiplication.

2. The functions $\log_e x$ and e^x

For $t > 0$ we have $\int t^n dt = t^{n+1}/(n+1)$, where $n \in \mathbb{Z}, n \neq -1$. What is the value of this integral when $n = -1$? The question is very natural. Let $f : R_+ \to R_+$ be the function given by $f(t) = 1/t$. Since f is continuous, by Cauchy's theorem[1], f is Riemann integrable on any compact interval $[a, b]$ in R_+. Define

$$L : R_+ \to R \text{ by } L(x) = \int_1^x f(t)\, dt.$$

Theorem 1: The function L is differentiable and $L'(x) = 1/x$.

Proof: Since f is continuous, the proof follows by the fundamental theorem of calculus [2].

Theorem 2: The function $L : R_+ \to R$ is a continuous isomorphism.

Proof: Continuity of L follows by Theorem 1. Since $f(t) > 0$ for $t > 0$, it follows by a property of Riemann integral [3] that L is injective. We will show that L is surjective. Let $y \in R$. Choose $\alpha < 0$ and $\beta > 0$ such that $\alpha < y < \beta$. For $x > 1$ and $n \in N$, $\int_1^x 1/t\, dt > \int_1^x 1/t^{1+1/n} dt = n[1 - 1/x^{1/n}]$. Choose $n > 2\beta$ and $x_2 = 2^n$ we get $L(x_2) = \int_1^{x_2} 1/t\, dt > n[1 - 1/2] = n/2 > \beta$. Similarly we can find $x_1 \in R_+$ such that $L(x_1) < \alpha$. Therefore by intermediate value theorem [4] there exists $x \in R_+$ such that $L(x) = y$. This proves that L is surjective. Next we will show that L as a homomorphism. Let $a \in R_+$ be arbitrary but fixed. By chain rule we have $d/dx(L(ax)) = (1/ax)\ a = 1/x$. Therefore $L(ax) = L(x) + c$ This gives $L(a) = L(1) + c = c$. Therefore $L(ax) = L(x) + L(a)$. Since a is arbitrary we have L is homomorphism. This completes the proof.

Let $b > 0, b \neq 1$ be a fixed positive real number and let

[1] Every continuous real valued function on $[a, b]$ is Riemann integrable.

[2] If f is a continuous real valued function on an interval $[a, b]$, then the function $F(x) = \int_a^x f(t)\, dt, x \in [a, b]$ is differentiable and $F'(x) = f(x)$, for every $x \in [a, b]$.

[3] If f is a continuous real valued function on $[a, b]$ and $f(x) > 0$, for every $x \in [a, b]$ then $\int_x^y f(t)\, dt > 0$, for $a \leq x < y \leq b$.

[4] If f is a continuous real valued function on a connected subset of R, then f takes every value between any two values it assumes.

$g : R \to R_+$ be a function given by $g(x) = b^x$. Clearly g is a continuous isomorphism.

Theorem 3: $\{b^x : b > 0, b \neq 1\}$ are the only continuous isomorphisms from R onto R_+.

Proof: Let g be a continuous isomorphism from R onto R_+. Using the homomorphism of g, it can be easily shown that for any rational number $r, g(rx) = g(x)^r$ Since g is continuous and rationals are dense in reals, it follows that $g(sx) = g(x)^s$ for any real number s. In particular we have $g(x) = g(x.1) = g(1)^x = b^x$, where $b = g(1)$. Since the range of g is $(0, \infty)$ we must have $b > 0$ and $b \neq 1$. This completes the proof.

Let $g : R \to R_+$ be a continuous isomorphism given by $g(x) = b^x, b > 0, b \neq 1$. The inverse of g is given by $g^{-1}(y) = x$ if $y = b^x$. We denote $g^{-1}(y)$ by $\log_b y$.

Theorem 4: $\{\log_b y : b > 0, b \neq 1\}$ are the only continuous isomorphisms from R_+ onto R.

Proof: Suppose $h(y) = \log_b y$ for some $b > 0, b \neq 1$. Then $h = g^{-1}$, where $g : R \to R_+$ is given by $g(x) = b^x$. Since g is an isomorphism, g^{-1} is also an isomorphism. We will show that g^{-1} is continuous. Let $y \in R_+$ be arbitrary but fixed. Choose $\epsilon > 0$ such that $[y - \epsilon, y + \epsilon]$ lies inside R_+. Since g is bijective, there exists $c, d \in R$ such that $g(c) = y - \epsilon$ and $g(d) = y + \epsilon$. Clearly g is strictly decreasing if $0 < b < 1$ and strictly increasing if $b > 1$ and hence it is injective. Since g is continuous, by intermediate value theorem g maps $[c, d]$ onto $[y - \epsilon, y + \epsilon]$. By open mapping theorem for (compact) metric space [5] $g^{-1} : [y - \epsilon, y + \epsilon] \to [c, d]$ is continuous. Therefore g^{-1} is continuous at y. Since $y \in R_+$ is arbitrary, g^{-1} is continuous on R_+. Therefore h is a continuous isomorphism.

[5] If f is a continuous injective map from a compact metric space X onto a metric space Y then f^{-1} is continuous.

Now we show that every continuous isomorphism from R_+ onto R is of the form $\log_b y$ for some $b > 0, b \neq 1$. Let $\phi : R_+ \to R$ be a continuous isomorphism. Clearly

ϕ^{-1} is an isomorphism. By open mapping theorem for (compact) metric spaces, it follows that ϕ^{-1} is continuous. Hence by Theorem 3 we have $\phi^{-1}(x) = b^x$ for some $b > 0, b \neq 1$. Thus $\phi(y) = (\phi^{-1})^{-1}(y) = \log_b y$. This completes the proof.

By Theorem 2 and Theorem 4 it follows that $L(x) = \log_b x$ for some $b > 0, b \neq 1$. We denote this b by e. We find the value of e. As a first step we prove the following:

Corollary 5: $\log_e(x^r) = r\log_e x$ for all $r \in R$, $x \in R_+$.

Proof: Let $g = L^{-1}$. We have $g(L(x^r)) = x^r$ Since $g(sx) = g(x)^s$ we have $g(rL(x)) = (g(L(x)))^r = x^r$ Since g is injective we have $L(x^r) = rL(x)$.

Theorem 6: The value of e is $\lim_{t\to 0}(1+t)^{1/t}$.

Proof: For $x > 0$ and for $h \in R$ with $h \neq 0, -x < h$, by Corollary 5, we have $(1/h)[\log_e(x+h) - \log_e x] = (1/x)\log_e(1+t)^{1/t}$ where $t = h/x$. From this and the continuity of the function $\log_e y$ it follows that $d/dx \log_e x = 1/x \log_e c$ where $c = \lim_{t\to 0}(1+t)^{1/t}$. Since $d/dx \log_e x = 1/x$ this implies that $\log_e c = 1$. Moreover by the definition of $\log_e x$ we have $\log_e(e^x) = x$. In particular $\log_e e = 1$. Since the function $\log_e x$ is injective, it follows that $e = c$. This completes the proof.

Theorem 7: $d/dx e^x = e^x$ and $e^x = \Sigma x^n/n!$.

Proof: Now $d/dx \log_e x = 1/x$. Define $f_1 : R_+ \to R$ by $f_1(x) = \log_e x$. Then $f_1'(x)1/x > 0$. Now f_1 has a inverse function g_1 given by $g_1(y) = e^y$. By inverse mapping theorem ([2], Ex: 2, pp.114), $g_1(f_1(x)) = 1/f_1(x)$. Therefore $g_1(y) = x = e^y$. This proves $d/dy(e^y) = e^y$. The second result follows by Taylor series formula.

3. The Functions e^z and $\log e^z$

In this section, we extend the function $e^x : R \to R_+$ to a continuous homomorphism from $\mathbb{C}$ into $\mathbb{C}^*$. Let $g : \mathbb{C}^* \to \mathbb{C}$ be a continuous homomorphism. Then

$0 = g(1) = g[(-1)(-1)] = g(-1) + g(-1)$, which implies that $g(-1) = 0$ and hence g is not injective. This shows that there is no continuous isomorphism from $\mathbb{C}^*$ onto $\mathbb{C}$. Hence there is no continuous isomorphism from $\mathbb{C}$ onto $\mathbb{C}^*$ Moreover, in the discussion of the function e^x, we considered the group $(R_+, \cdot)$ instead of the group $(R^* \cdot)$ because continuous image of a connected set is connected and homomorphism preserves identity. So we look for continuous homomorphism from $\mathbb{C}$ into $\mathbb{C}^*$ which extends e^x. Let $F : \mathbb{C} \to \mathbb{C}^*$ be any continuous function satisfying the conditions:

(i) $F(z_1 + z_2) = F(z_1)F(z_2), z_1, z_2 \in \mathbb{C}$

(ii) $F(x) = e^x, x \in R$.

Since $F(z) = F(x + iy) = F(x)F(iy) = e^x F(iy)$ it is enough if we know the value of $F(iy)$ for $iy \in I = \{iy : y \in R\}$, the imaginary axis, to know the value of $F(z)$.

Theorem 8: $\{f_w : w \in \mathbb{C}^*\}$, where $f_w(iy) = w^y$, are the only continuous functions from I into $\mathbb{C}^*$ which satisfy $f_w(iy_1 + iy_2) = f_w(iy_1)f_w(iy_2)$.

Proof: Suppose f is a continuous function from I into $\mathbb{C}^*$ which satisfies the condition given in the hypothesis. Take $w = f(i)$. Then, as in the proof of Theorem 3, it can be easily shown that $f(iy) = f(i)^y = w^y$. Conversely, if f is a function from I into $\mathbb{C}^*$ given by $f(iy) = w^y$ for some $w \in \mathbb{C}^*$, it is easy to verify that f satisfies the conditions given in the hypothesis.

Theorem 9: $\{F_w : w \in \mathbb{C}^*\}$, where $F_w(z) = F_w(x + iy) = e^x f_w(iy)$, are the only continuous functions from $\mathbb{C}$ into $\mathbb{C}^*$ which satisfy conditions (i) and (ii) above.

Proof: The proof follows by Theorem 8 and the above discussion.

Now we study the functions F_w in detail. Let F_w be a function as in Theorem 9. Then $w = r(\cos\alpha + i\sin\alpha)$ for some $r > 0$ and some $\alpha \in (-\pi, \pi)$. The reason for taking the interval $(-\pi, \pi)$ instead of the interval $[0, 2\pi]$ will be justified in Theorem 12.

Case (i) Suppose $\alpha = 0$ and $r = 1$. Then $F_w(z) = F_w(x + iy) = e^x$.

Case (ii) Suppose $\alpha = 0$ and $r \neq 1$. Then $F_w(z) = F_w(x + iy) = e^x r^y$.

Case (iii) Suppose $\alpha \neq 0, \alpha \neq 1$ and $r = 1$. Then $F_w(z) = F_w(x + iy) = e^x(\cos \alpha y + i \sin \alpha y)$.

Case (iv) Suppose $\alpha \neq 0, \alpha \neq 1$ and $r \neq 1$. Then $F_w(z) = F_w(x + iy) = e^x r^y(\cos \alpha y + i \sin \alpha y)$.

Case (v) Suppose $\alpha = 1$ and $r = 1$. Then $F_w(z) = F_w(x + iy) = e^x(\cos y + i \sin y)$.

It is easy to verify that F_w as in cases (i)-(iv) are not even harmonic and hence not analytic. However the following theorem shows that F_w as in case (v) is analytic. Since e^x is real analytic, it is natural to look for a complex analytic extension of e^x. So the extension F_w as in case (v) is the perfect choice. Let $\gamma = \cos 1 + i \sin 1$. We denote the corresponding F_γ by E. Then $E(z) = E(x + iy) = e^x(\cos y + i \sin y)$.

Theorem 10: The function $E : \mathbb{C} \to \mathbb{C}^*$ is analytic, $E(z) \neq 0, E'(z) = E(z)$ for $z \in \mathbb{C}$, and $E(z) = \Sigma z^n/n!$.

Proof: Clearly E is continuous and satisfies Cauchy–Riemann equation and hence it is analytic. Using the relation $E'(z) = u_x + iv_x$, where u and v are the real and the imaginary parts of E, it can be easily seen that $E'(z) = E(z)$. Now by Theorem 2.8, Chapter IV in [1] it follows that $E(z) = \Sigma x^n/n!$. The proof is over.

Note that the derivative of $E(z)$ is itself and the power series expansion of $E(z)$ is similar to that of e^x (Theorem 7). This is an unexpected beauty of the function $E(z)$. Because of this similarity we normally denote $E(z)$ by e^z. However this is just a notation, this is nothing to do with the Euler number e.

Let $T = \{w \in \mathbb{C}^* : |w| = 1\}$ and for each $n \in \mathbb{Z}$ let $I_n = \{iy : y \in ((2n-1)\pi, (2n+1)\pi)\}$. Clearly I_n's are disjoint and their union is I. For each $n \in \mathbb{Z}$ let f_n be the restriction of f_γ to I_n. Since $\cos y$ and $\sin y$ are 2π-periodic, $f_\gamma : I \to T$ is not injective but it is surjective. However $f_n : I_n \to T$ is bijective. Let $S_n = \{x + iy : x \in R, iy \in I_n\}$. Clearly S_n's are disjoint and their union is $\mathbb{C}$. For each $n \in \mathbb{Z}$ let E_n be the restriction of E to S_n. Since f_γ is not injective, $E : \mathbb{C} \to \mathbb{C}^*$ is also not injective but is surjective. However $E_n : S_n \to \mathbb{C}^*$ is bijective. Let $L_n : \mathbb{C}^* \to S_n$ be the inverse of E_n. Let $S_n^0 = \{x + iy : x \in R, y \in ((2n-1)\pi, (2n+1)\pi)\}$ be the interior of S_n and let $G = E_n(S_n^0)$. Clearly $G = \mathbb{C}^* \backslash \{u \in R : u < 0\}$. Normally $L_0(z)$ is denoted by $\log_e z$.

Theorem 11: Each L_n is continuous except at points $u, u < 0$ and it is analytic on G. Moreover $L_n'(w) = 1/w$ at each $w \in G$.

Proof: Suppose $w \in \mathbb{C}^*$ Since E_n is bijective there exists a unique $z = u + iy \in S_n$ such that $E_n(z) = w$. This implies that $x = \log_e |w|$ and $y = \arg(w) + 2n\pi$. Since $\arg(w)$ is continuous except at points $u, u < 0, L_n$ is also continuous except at points $u, u < 0$. Now the restriction of E_n on S_n^0 is bijective, and by Theorem 10, E_n is differentiable on S_n^0 and $E_n'(z) \neq 0$ for all $z \in S_n^0$. Therefore, by Proposition 2.20, Chapter III in [1], L_n is analytic on G. Moreover, by the same Proposition, $L_n'(w) = 1/w$.

Theorem 12: For any $z \in \mathbb{C}^* \backslash \{u \in R : u < 0\}$ we have $\int_\gamma 1/w\,dw = \log_e z$, where γ is any rectifiable path in $\mathbb{C}^* \backslash \{u \in R : u < 0\}$ joining 1 and z.

Proof: The proof follows by the complex analogue of the fundamental theorem of calculus (Proposition 1.18, Chapter IV in [1]).

Suggested Reading

[1] J B Conway, *Functions of one Complex Variable*, 2nd ed., Springer-Verlag, New York, 1973.

[2] W Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill Book Company, New York, 1976.

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