Think It Over

This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

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Solution to Tangles of Mats and Maths of Rectangles

1. *Let G be a finite group all of whose proper subgroups are abelian. Prove that G is solvable.*

Proof: We prove it by induction on $|G| = n$. For $n = 1$, it is trivial. Let it be true for all $n < m$, and let G be a group of order m satisfying the given property. If G has a nontrivial, proper normal subgroup N then N is abelian, and G/N has the given property. As $O(G/N) = \frac{O(G)}{O(N)} < m$, by the induction hypothesis, G/N is solvable. As N (being abelian) is also solvable, G is solvable. Hence, we may assume that G has no nontrivial, proper normal subgroup. In particular, $Z(G)$ (centre of G) $= \{e\}$.

On $G \backslash \{e\}$, define a relation $\sim$ as follows:

$$x \sim y \Leftrightarrow xy = yx.$$

Obviously $\sim$ is reflexive and symmetric. For transitivity, we first observe that any two noncommuting elements generate G, since if they generate a proper subgroup of G, it would have to be abelian, contradicting the fact that they do not commute.

So, if $x \sim y$ and $y \sim z$ but $x \not\sim z$, then x and z generate G. But y commutes with x and z and so $y \in Z(G)$, a contradiction, since G has trivial centre.

Hence, consider the equivalence classes of $\sim$. We claim that conjugation gives an action on these equivalence classes. If A is an equivalence class and $x \in G$, then $xAx^{-1} = B$ is a set of commuting elements in $G\backslash\{e\}$ and so, B is contained in an equivalence class C. If $g \in C$, then g commutes with $B = xAx^{-1}$; so $x^{-1}gx \in A$ i.e., $g \in xAx^{-1} = B$. Hence $C = B$ and conjugation by G is indeed an action on the set of equivalence classes.

If A is an equivalence class, we claim that its stabiliser stab $A = A \cup \{e\}$. If $b \notin A \cup \{e\}$ stabilises A, then b does not commute with elements of A; therefore A and b generate G. But A and $\{b\}$ are contained in stab $A \Rightarrow$ stab $A = G \Rightarrow A \cup \{e\}$ is normal in G. If $A \cup \{e\} = G$, we are done since $A \cup \{e\}$ is abelian. Otherwise, we have a proper normal subgroup of G, a contradiction. Thus stab $A = A \cup \{e\}$. If $|A \cup \{e\}| = k$, orbit of A has $\frac{O(G)}{O(A\cup\{e\})} = \frac{m}{k}$ elements, and these equivalence classes contain $\frac{m}{k}(k-1) \geq \frac{m}{2}$ distinct elements (different from e) of G. If one more orbit were there, we would have $\geq \frac{m}{2} + \frac{m}{2} = m$ elements, so we have no room for e. So this is the only orbit. But then $\frac{m}{k}(k-1) + 1(\text{for } e) = O(G) = m \Rightarrow m = k$, so $G = A \cup \{e\}$ which is abelian and hence solvable.

2. *Let G be a finite group such that for each positive integer n, G has at the most n elements satisfying $g^n = 1$. Prove that G must be cyclic.*

Proof I: Let $|G| = m$. We claim that the number $N(d)$ of elements $\{g : O(g) = d\}$ is $\varphi(d)$ or zero for each d dividing m.

Suppose $O(g) = d$ for some $g \in G$. Then $\{1, g, g^2,$ $g^{d-1}\}$ are distinct elements whose dth power is 1. Since there are d of these, no element h outside this set can satisfy $h^d = 1$. Clearly, $O(g^e) = d \Leftrightarrow (d, e) = 1$. Therefore, there are exactly $\phi(d)$ elements of order d, proving the claim.

But $\sum_{d/m} N(d) = m$ as each $g \in G$ has some order dividing m. Also $\sum_{d/m} \varphi(d) = m$. So $N(d) = \phi(d)$ and there are exactly $\varphi(d)$ elements of order d in G for each d dividing m. In particular there are $\phi(m)$ elements of order m.

Proof II: Any subgroup H of G is the unique subgroup of order $O(H)$ by the hypothesis. In particular, all p-Sylow subgroups are unique; therefore they are normal. If P,Q are the p-Sylow subgroup, and the q-Sylow subgroup respectively, where $p \neq q$, then for $x \in P, y \in Q$, we have $xyx^{-1}y^{-1} \in P \cap Q$ as both P, Q are normal. However $P \cap Q = \{1\}$ since its order must both be a power of p and a power of q. Thus, P, Q mutually commute and it suffices to show that the p-Sylow subgroups are cyclic. If $O(P) = p^n$, then the number of elements of order $< p^n$ is at the most $1+p+p^2+\cdots+p^{n-1}$, which is evidently $< p^n$. Thus, P has a generator.

3. *If* $p(x)$ *and* $q(x)$ *are polynomials, with* $\deg p(x) \leq \deg q(x) - 2$ *and* $q(x)$ *has all its roots* $\alpha_1, \ldots, \alpha_n$ *distinct, then prove*

$$S = \sum_{i=1}^{n} \frac{p(\alpha_i)}{q'(\alpha_i)} = 0.$$

Proof: Clearly, we may assume that q is monic i.e., $q(x) = \prod_{i=1}^{n}(x - \alpha_i)$. Then $q'(x) = \sum_i \prod_{j \neq i}(x - \alpha_j)$ so that $q'(\alpha_i) = \prod_{j \neq i}(\alpha_i - \alpha_j)$.

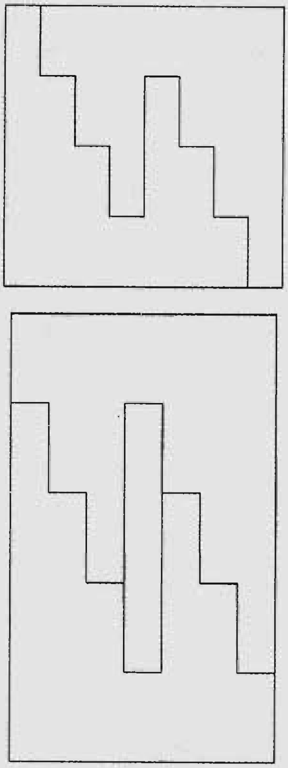
We need to show that $S := \sum_i \frac{p(\alpha_i)}{\prod_{j \neq i}(\alpha_i - \alpha_j)} = 0$.
We claim that actually

$$p(x) = \sum_i p(\alpha_i) \prod_{j \neq i} \frac{x - \alpha_j}{\alpha_i - \alpha_j}.$$

To see this, consider $h(x) = \sum_i p(\alpha_i) \prod_{j \neq i} \frac{x - \alpha_j}{\alpha_i - \alpha_j} - p(x)$.

Now $h(\alpha_i) = 0$ for all $i = 1, \ldots, n$. But $h(x)$, clearly, has degree atmost $(n-1)$. Thus, $h(x)$ must be the zero polynomial. In other words, the claim that $p(x) =$

Answer to 5:



$\sum_i p(\alpha_i) \prod_{j \neq i} \frac{x-\alpha_j}{\alpha_i - \alpha_j}$ is proved. As $p(x)$ has degree at the most $n-2$, the right hand side of the above expression must have its coefficient of x^{n-1} (which is just S) to be zero.

4. *Start with a square of side* 1. *Right on its top, add a unit square. Next, to the right of this rectangle, adjoin a rectangle of the same height (2 in this case) and of unit area. At this stage, the resultant rectangle has height* 2 *and width* 3/2. *Proceed in this manner, adjoining alternately rectangles to the top (with the same base as in the previous stage) and to the right (with the same height as the previous stage), taking care that each time the added rectangle has unit area. What is the limit of the ratio of height to the width?*

Proof: Let s_n and t_n be total width and height at a stage when there are $2n$ rectangles. It is clear that the initial values are

$$\begin{array}{rclrcl} s_1 &=& 1 & t_1 &=& 2 \\ s_2 &=& 3/2 & t_2 &=& 8/3. \end{array}$$

The description translates to the recurrence equations

$$\begin{array}{rclcrcl} s_{n+1} - s_n &=& 1/t_n & \Rightarrow & t_n s_{n+1} - t_n s_n &=& 1 \\ t_{n+1} - t_n &=& 1/s_{n+1} & \Rightarrow & t_{n+1} s_{n+1} - t_n s_{n+1} &=& 1. \end{array}$$

Adding,

$$t_{n+1} s_{n+1} - t_n s_n = 2.$$

Since $s_1 t_1 = 2$ and $s_2 t_2 = 4$, we have $s_n t_n = 2n \ \forall \ n \Rightarrow t_n = 2n/s_n$.

So $\frac{s_{n+1}}{s_n} = 1 + \frac{1}{2n}$ which gives easily $s_{n+1} = \frac{(2n+1)!}{2^{2n}.(n!)^2}$.

We want

$$l := \lim_{n \to \infty} t_n / s_n = \lim \frac{2n}{{s_n}^2} = \lim 2n \ \frac{2^{4n} n!^4}{(2n+1)!^2}.$$

Hence $l = \lim 2n \ \frac{2^{4n} (\sqrt{2\pi})^4 n^{4n+2} e^{-4n}}{(\sqrt{2\pi})^2 (2n+1)^{4n+3} e^{-4n+2}}$ by Stirling's formula.

Finally, $l = \frac{\pi}{2} e^{-2} \lim (1 + \frac{1}{2n})^{4n+2} = \pi/2$.

5. *Given two pieces of mat of dimensions* 8×8 *and* 1×6 *respectively, how can the* 8×8 *piece be cut into exactly two pieces so as to make the resultant three pieces fit to form exactly a* 10×7 *mat?*

ERRATA

Vol. 6, No.8, August 2001

Title: Numeracy for everyone
Page 9, line 28:

Wheels is a novel by Arthur Hailey not Irving Wallace as mentioned in the article.

Vol. 6, No.9, September 2001

Title: The Importance of Being Ignorant
Page 13, Box 2.

Conditional probability of a given that b has occurred $= p(a|b)$

$$= \frac{\text{area of } C}{\text{area of } B} = \frac{p(a,b)}{p(b)}$$

Hence, $p(a,b)=p(a|b)=p(a|b).p(b)$. Similarly, $p(a,b)=p(b|a).p(a)$.

Page 18, Figure 4. Picture of radio emission from the galaxy M81 made with the Giant Metrewave Radio telescope, Khodad, at a wavelength of half a metre. The image on the right was obtained from that on the left using extra prior information. As a result, radio emission from a supernova explosion which was first seen in 1993 has become visible. (Thanks to Poonam Chandra, Alak Ray (TIFR) and Sanjay Bhatnagar (NCRA-TIFR) from whose ongoing work this example is taken).