

# Entropy à la Boltzmann

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## Introduction

A discussion of entropy after the life and times of Boltzmann must of necessity be anachronistic [1]. Entropy historically was introduced in the early study of thermodynamics, a science which brought together the previously unrelated mechanical and thermal energies. Thermodynamics was robust enough to survive the two great revolutions of the early twentieth century – the theory of relativity and the quantum theory. Consequently, it might seem natural to stay with thermodynamics and introduce entropy from that standpoint. Thermodynamics is a practical science based on physical arguments, and has its genesis in efforts to understand the efficiencies of various engines, which were devices that converted thermal energy to mechanical energy. However, it is also true that thermodynamics is a fairly abstract [2] science and the introduction of entropy from the thermodynamic point of view cannot be immediately visualized via a physical picture. Paradoxically it is the Boltzmann picture, which was little understood and even less believed in Boltzmann's lifetime, which is somewhat more physically accessible. This is to a large extent because the molecular picture to which Boltzmann appealed constantly is today much better understood and established beyond doubt. In contrast, the molecular picture was not universally accepted in the latter half of the nineteenth century and it was impossible for Boltzmann's contemporaries to appreciate his point of view. However, after the developments of the twentieth century, it is more normal to approach entropy from the Boltzmann view [3] and that is what we will try to do.

As it turns out, entropy forms the natural bridge between the thermodynamic description of the world and the microscopic description.

As it turns out, entropy forms the natural bridge between the thermodynamic description of the world and the microscopic description. The thermodynamic description of a system involves a few macroscopic parameters. If we consider a cylinder of gas as our system (a practical and popular thing to do), then its thermodynamic state is characterized by variables like pressure ( $P$ ), volume ( $V$ ), and temperature ( $T$ ). For a fixed number of moles of the gas (i.e. for a fixed mass), the above variables are not generally independent. They are related by a relation known as the equation of state.

### Microscopic Description

We now need to understand the microscopic state of our system. To do this we need to know how many molecules of the gas are present in the cylinder. Let us say this number is  $N$ . For the present purpose, we will ignore all internal structure of the molecule and take it to be the smallest constituent of the gas. First, we shall assume that the motion of the molecule is governed by classical mechanics. A particular molecule, say the  $i^{\text{th}}$  one has at a given instant, the velocity  $\vec{v}_i$  and position  $\vec{r}_i$ . If the mass of the molecule is  $m$  then Newton's law could be written as  $m\ddot{\vec{r}}_i = \vec{f}_i$ , where  $\vec{f}_i$  is the force acting on the  $i^{\text{th}}$  molecule. It is possible to write this as two first order equations.

$$\begin{aligned}\dot{\vec{r}}_i &= \frac{\vec{p}_i}{m} \\ \dot{\vec{p}}_i &= \vec{f}_i.\end{aligned}\quad (1)$$

For the  $i^{\text{th}}$  molecule in three dimensional space the above equations actually represent 6 first order differential equations. If we supply the values of the 6 variables (three components of  $\vec{r}_i$  and three components of  $\vec{p}_i$ ) at time  $t=0$ , then (1) can be integrated to yield  $\vec{r}_i$  and  $\vec{p}_i$  at all subsequent times. Thus the motion of the  $i^{\text{th}}$  molecule can be known completely for all time once the six initial conditions are given. At each stage, the initial

conditions are updated, which is to say, the six variables are known at each instant and these six variables (three coordinates and three momenta) constitute the state of the particle. If this information is provided for all the molecules of the gas, then the microscopic state of the gas is said to be specified. Thus the microscopic state of the system is defined by specifying  $6N$  variables.

The number of molecules in a macroscopic system is of the order of  $10^{23}$  (Avogadro number). Consequently, the specification of the microscopic state of the gas is rather difficult. Further it is intuitively obvious that the microstate will keep on changing continuously. If we tag the different molecules, it is clear that the different molecules will be at different points of the space inside the cylinder at different instants of time. However, the macroscopic variables – the pressure, temperature, etc. do not seem to change at all. Corresponding to a given macroscopic state there must then be millions and millions of microscopic states. The number of microstates corresponding to a given macroscopic state is called the multiplicity of the macrostate. The imagined space with  $6N$  axes representing the  $6N$  momenta and coordinates of the  $N$ -particles is called the phase space of the system. A particular microstate of the system is a point in the phase space. Since a huge number of microstates correspond to a given macrostate, it is reasonable to think of the macrostate as a volume in the phase space.

We now want to calculate the multiplicity of a given macrostate. It is somewhat surprising that to do this simple counting, we need to take recourse to quantum mechanics. As a result we will make a short digression to discuss how the specification of the state of a given particle (remember in classical mechanics this specification requires the three coordinates  $x, y, z$  and the three momenta) has to be modified in this new mechanics. The central paradigm of quantum mechanics is that it is not possible to specify the coordinate ( $x$ ) and the



corresponding momentum ( $p_x$ ) simultaneously with arbitrary accuracy. This new dynamics is governed by Heisenberg's uncertainty principle which states that if  $\Delta x$  and  $\Delta p_x$  are the uncertainties (standard deviations) associated with the simultaneous measurement of the  $x$ -coordinate and the  $x$ -momentum then

$$\Delta x \Delta p_x \geq \frac{h}{4\pi}, \quad (2)$$

where  $h$  is a fundamental constant which is known as Planck's constant and is approximately  $6.6 \times 10^{-27}$  erg sec. Similar relations hold for  $y$  and  $p_y$ ,  $z$  and  $p_z$ . However, there is no such bar to simultaneous, arbitrarily precise measurements of  $y$  and  $p_x$  or in general for one component of the position coordinate and a *different* component of the momentum.

Now imagine a particle moving in one dimension i.e. along a line. We constrain the motion to lie within a length  $L$  of the line. Let us suppose, the velocity cannot exceed a certain maximum value  $v_{\max}$  (this cannot be strictly true, but the probability of having a velocity  $v > v_{\max}$  can be made extremely small). The particle can move to the right or left and so the velocity can range from  $-v_{\max}$  to  $+v_{\max}$ . The region of phase space accessible to the particle (the phase space is two dimensional, the axes being  $x$  and  $p_x$ ) is the rectangle shown in *Figure 1*.

Let us try to specify the state of a particle as accurately as we can in this figure. Classically, the state is a point, but in a mechanics obeying (2), the best that one should be able to do is to specify the state within a rectangle of area  $\frac{h}{4\pi}$ . Actual calculation shows that the unit is  $h$  rather than  $\frac{h}{4\pi}$ . In *Figure 1*, the shaded boxes have area  $h$  rather than  $\frac{h}{4\pi}$ . Non-overlapping boxes are completely independent microstates and hence the number of microstates in the macrostate (the area is  $2mv_{\max}L$ ) is  $\frac{2mv_{\max}L}{h} = \frac{2\sqrt{2mEL}}{h}$ , where  $E = \frac{1}{2}mv^2$  is the kinetic energy.



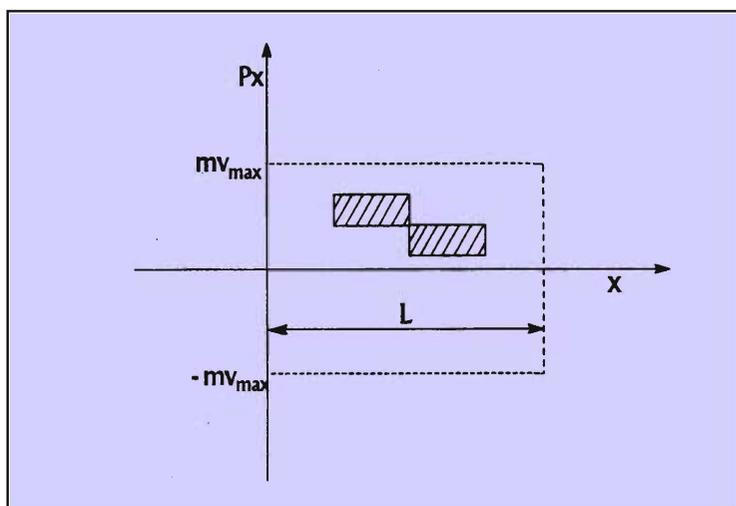


Figure 1. Phase space diagram.

In three dimensions, the line becomes a box of volume  $V$  and the number of microstates will be proportional to  $(2mE)^{3/2}V/h^3$ , which is essentially the formula for the line raised to the third power. The exact answer including the constant in front is that the number of microstates available for a particle confined in a three dimensional box of volume  $V$  and having energy less than or equal to  $E$  is  $\frac{4\pi}{3}(2mE)^{3/2}V/h^3$ . Now if we have  $N$  particles in the box, then the phase space becomes  $6N$  dimensional instead of 6 dimensional as it was for the single particle. From 2-dimensional to 6-dimensional space the factor  $\frac{\sqrt{2mE}}{h}$  had to be raised to the power 3. It should be apparent that from 2-dimensional to  $6N$ -dimensional space, the relevant power would be  $3N$  and the multiplicity of macrostates would be proportional to  $(2mE)^{3N/2}V^N/h^{3N}$ . The constant of proportionality can only emerge from an explicit calculation. The indistinguishability of the particles would reduce the final answer by a factor of  $N!$  because permuting the particles would not lead to a new state. Finally then, for  $N$  particles in a volume  $V$  with total kinetic energy at most  $E$ , the number of microstates  $\Omega$  is given by the formula

$$\Omega = \frac{1}{N!} S_{3N} \left[ \frac{(2mE)^{3/2}V}{h^3} \right]^N, \quad (3)$$

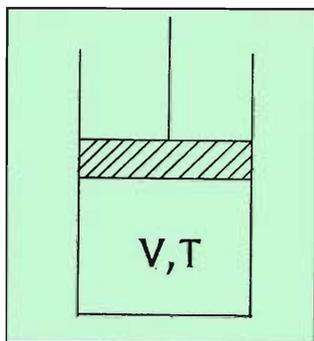


Figure 2. Gas in a cylinder with a movable piston.

where  $S_{3N}$  is the volume of a  $3N$ -dimensional sphere of radius unity.

### Connecting the Micro and the Macro

We now take our (ideal) gas to occupy a volume  $V$  at a temperature  $T$  in a cylinder whose top end is a movable piston as shown in Figure 2.

We supply a very small quantity  $\Delta Q$  of heat to the gas very slowly, i.e. sufficiently slowly for it to remain in thermal equilibrium with its environment i.e. it maintains its temperature  $T$ . The heat causes the volume to increase at constant temperature (isothermal) and the gas does work by pushing the piston. Since this is an ideal gas, its internal energy depends only on temperature, not on volume. Thus the entire effect of the heat supplied is to produce work and we can write

$$\text{Work done by the gas} = \text{Heat supplied to it} = \Delta Q. \quad (4)$$

Let us now exploit this simple fact to understand heat in terms of the number of microstates we discussed in the previous section. The work done by the gas is clearly  $P\Delta V$  if  $P$  is the pressure when the temperature is  $T$  and volume  $\Delta V$ . If the gas is taken to be ideal then, as you know from high-school physics,

$$PV = NkT \quad (5)$$

where  $k$  is a constant and, consequently,

$$\Delta Q = P\Delta V = NkT \frac{\Delta V}{V}. \quad (6)$$

This is the macroscopic view. In the microscopic view, as the gas has isothermally changed its volume from  $V$  to  $V + \Delta V$  the multiplicity of the state will change. If we look at (3), we note that the multiplicity changes because of the change in  $V$ . The other variables  $E$  and  $N$  are unchanged as no molecules are taken in or out and

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the temperature is held constant. If  $\Omega_i$  is the multiplicity when the volume is  $V$  and  $\Omega_f$  is the multiplicity when the volume is  $V + \Delta V$ , then from (3)

$$\frac{\Omega_f}{\Omega_i} = \left(1 + \frac{\Delta V}{V}\right)^N \quad (7)$$

Taking logarithms and expanding in the small variable  $\frac{\Delta V}{V}$  to the lowest order,

$$\ln \Omega_f - \ln \Omega_i = N \ln\left(1 + \frac{\Delta V}{V}\right) \simeq N \frac{\Delta V}{V}. \quad (8)$$

Comparing with (6), we find

$$\frac{\Delta Q}{T} = k \ln \Omega_f - k \ln \Omega_i. \quad (9)$$

We have thus connected  $\Delta Q$ , a quantity defined in terms of *macroscopic* measurements, with  $\Omega$ , which measures the number of *microstates*. Let us now *define* the entropy  $S$  as Boltzmann did:

$$S = k \ln \Omega. \quad (10)$$

It should be clear from (3) that  $S$  is extensive. From (9), we can relate the Boltzmann definition to the historical approach. For the quasistatic process described, the entropy as defined in (9) and (10) obeys

$$\frac{\Delta Q}{T} = \Delta S, \quad (11)$$

and this provides a completely macroscopic picture of entropy. If a quasistatic process supplies a  $\Delta Q$  amount of heat to a system of fixed temperature  $T$  then the quantity  $\frac{\Delta Q}{T}$  in the limit of  $\Delta Q \rightarrow 0$  is a perfect differential which is denoted by  $dS$ , where  $S$  is called the entropy.

We now return to the situation shown in *Figure 2* and consider the system to be thermally isolated i.e. no heat

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is allowed to leave or enter the system. The piston is now slowly drawn out, sufficiently slowly that equilibrium is maintained all along. Since at every step of this process  $\frac{\Delta Q}{T} = \Delta S$ , we conclude that entropy cannot change in the process since  $\Delta Q = 0$  throughout. The relation between entropy and multiplicity given by (10) now asserts that the multiplicity cannot change – increasing the volume increases the multiplicity as we have just seen. To hold the multiplicity constant (4), now tells us that the temperature must decrease and this must happen in a manner such that  $T^{3/2}V = \text{constant}$ . This relation between temperature and volume for an ideal monatomic gas is experimentally verified. It should be realized that this is the relation that follows from the laws of thermodynamics and is expressed in the form  $TV^{\gamma-1} = \text{constant}$  for an adiabatic expansion. Here  $\gamma$  is the ratio of specific heat at constant pressure to the specific heat at constant volume. It is  $5/3$  for an ideal monatomic gas and consequently this relation is equivalent to what we have on the basis of (4) and (10). We thus conclude that

- If an amount of heat  $Q$  is supplied to a system at constant temperature  $T$ , its entropy increases by  $\frac{Q}{T}$
- If a system is kept thermally isolated while it allowed to change its macroscopic parameters in a quasistatic fashion, then entropy is unchanged and this entails a definite relation between volume and temperature or volume and pressure, etc.

The processes that we have described so far occur reversibly which means that if we reversed the conditions, then the system would retrace its path and eventually end up from where it started. It should be noted that in both cases dealt with above, the entropy of the entire universe (i.e. the system together with the environment) remains unaltered. In the isothermal expansion

the entropy of the gas increases by  $\frac{\Delta Q}{T}$ , but the environment which is also at temperature  $T$  loses an amount of entropy  $\frac{\Delta Q}{T}$  and hence the entropy of the universe does not change. In the next situation, where no heat is exchanged, the total entropy is trivially unaltered. The entropy of the entire universe is unaltered in a reversible process.

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### Examples

Armed with (10), we can now examine entropy changes for a couple of familiar irreversible processes. The first one that we will study is the mixing of equal volumes ( $V$ ) of hot and cold ideal gas at temperatures  $T_1$  and  $T_2$ , respectively. After a sufficiently long time has elapsed, the final temperature of the mixed gas will be  $\frac{1}{2}(T_1 + T_2)$ . The initial entropy  $S_i$  is the sum of the hot and cold parts and we have from (10) and (4)

$$S_i = Nk \left[ \frac{3}{2} \ln T_1 + \ln \frac{V}{N} + C \right] + Nk \left[ \frac{3}{2} \ln T_2 + \ln \frac{V}{N} + C \right], \quad (12)$$

where  $C$  is a constant involving  $\pi, \hbar, k$ , etc. The final entropy, which is the entropy of  $2N$  molecules occupying volume  $2V$  is

$$S_f = 2Nk \left[ \frac{3}{2} \ln \frac{T_1 + T_2}{2} + \ln \frac{V}{N} + C \right]. \quad (13)$$

The change in entropy in the process is

$$S_f - S_i = \frac{3Nk}{2} \ln \frac{(T_1 + T_2)^2}{4T_1T_2} > 0 \quad (14)$$

since  $(T_1 + T_2)^2 > 4T_1T_2$ .

This process of mixing is irreversible. If we picture this process as placing a slab of gas at higher temperature  $T_1$  over a slab of gas at lower temperature  $T_2$ , then as time goes on the process of heat conduction brings the final



Figure 3. Free expansion.

temperature to the uniform value of  $\frac{1}{2}(T_1 + T_2)$ . If we go on waiting for a long time it never so happens that the upper slab will be at temperature  $T_1$  and the lower slab at temperature  $T_2$  again. The process that has occurred appears to be irreversible.

We now take up another example. A rectangular box shown in *Figure 3* is divided into two equal halves by means of a diaphragm. The two chambers have the same volume and the box is kept in thermal equilibrium with a heat bath such that its temperature always remains at a fixed value  $T$ . The left chamber is filled with gas while the right chamber is empty.

At a certain instant we remove the partition and wait. The gas after a while occupies the entire volume  $2V$ . The initial entropy  $S_i$  is

$$S_i = Nk[\ln T + \ln \frac{V}{N} + C] \tag{15}$$

while the final entropy is

$$S_f = Nk[\ln T + \ln \frac{2V}{N} + C] \tag{16}$$

leading to a change in entropy

$$\Delta S = S_f - S_i = Nk \ln 2 > 0. \tag{17}$$

Once again this process of the gas moving over from one chamber to occupy both the chambers is irreversible. This means that if we go on waiting it will never so happen that we will find the entire gas in the left chamber once again.

The above examples point to the fact that in a natural process entropy always increases. If we look at the phase space of the system of hot and cold gases we see that for the initial macrostate the allowed number of microstates is lower than the number of microstates for the final macrostate. In the mixing of hot and cold gases, the

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ratio of final and initial multiplicities is (for  $T_1 = 200K$  and  $T_2 = 300K$ )

$$\frac{\Omega_f}{\Omega_i} \simeq e^{N/16} \quad (18)$$

while for the diffusing gas, the ratio is  $e^{N \ln 2}$ . Since  $N \sim 10^{23}$ ,  $\Omega_f$  is immense compared to  $\Omega_i$ . We will come back to this feature shortly.

### Time-Reversal Symmetry, Ergodicity and Entropy: The Issue of Reversibility and Irreversibility

First, we would like to point out a somewhat problematic aspect of the time evolution of the systems described above. The key feature of the two time evolutions described above was irreversibility, i.e., the system evolved to a state from which it never came back to its initial state. The microscopic dynamics, however, is time reversible. This can be understood by looking at (1). If we reverse the signs of all the velocities (momenta) and change the sign of time, the equation remains unchanged. What this means is the following: if a set of  $N$  particles have the velocities  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N$  at time  $t$  and then evolve to have velocities  $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_N$  at time  $t+1$ , then if at time  $t+1$  we reverse the velocities to be  $-\vec{w}_1, -\vec{w}_2, \dots, -\vec{w}_N$  the velocities at time  $t+2$  will be  $-\vec{v}_1, -\vec{v}_2, \dots, -\vec{v}_N$  and the coordinates will be identical to what they were at time  $t$  which is essentially the state at time  $t$  (with velocities reversed). In other words, the system will return to its original state every now and then if we follow its microscopic dynamics. For example, if we consider the case of free expansion (*Figure 3*) this would imply that in the course of their motion the gas molecules, at certain times, would return to their original coordinates which would put all of them back in the left chamber and we would have a return to the starting condition. However, in practice, the gas never returns to the left chamber over any period of observation. The natural question is: since the microscopic

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To get a feeling for the dynamics involved, we recount briefly the subtleties of nonlinear dynamics, i.e., dynamics when the equation of motion is nonlinear. The dynamics is called *linear* if the dependent variable and its derivatives are present only in their first power, and nonlinear otherwise. The unforced simple harmonic oscillator  $\ddot{x} + \omega^2 x = 0$ , where  $\omega$  is the frequency of oscillation, is an example of a linear equation of motion, as is the damped driven simple harmonic oscillator  $\ddot{x} + k \frac{dx}{dt} + \omega^2 x = A \cos ft$ . The anharmonic oscillator  $\ddot{x} + \omega^2 x + x^3 = 0$  or the Duffing oscillator  $\ddot{x} - x^2 + x^3 = g \check{\cos} ft$ , on the other hand are nonlinear system because of the  $x^3$  term. The Van der Pol oscillator  $\ddot{x} + \dot{x}(x^2 - 1) + \omega^2 x = 0$  is also a nonlinear system because of the term  $\dot{x}x^2$ . The linear systems are always characterized by the fact that if we have two starting points in phase space (the  $x - \dot{x}$  space in each of the above cases) which are very close to each other, the points that evolve from these starting points always stay close to each other. We illustrate with  $\ddot{x} + \omega^2 x = 0$  with the initial conditions  $x = x_0, \dot{x} = v_0$  at  $t = 0$  the explicit solution is  $x = X_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$ . If the starting point is  $x = x_0 + \epsilon_1$  and  $\dot{x} = v_0 + \epsilon_2$  at  $t = 0$  then the solution is  $x = (x_0 + \epsilon_1) \cos \omega t + \frac{v_0 + \epsilon_2}{\omega} \sin \omega t$ . The separation of the two starting points in phase space is  $[\epsilon_1^2 + \epsilon_2^2]$ . At any subsequent time, the separation is  $[\epsilon_1^2 \cos^2 \omega t + \frac{\epsilon_2^2}{\omega^2} \sin^2 \omega t]^{1/2}$ , which remains small for all  $t$ . So neighbouring trajectories at one instant remain neighbouring at subsequent instants. The anharmonic oscillator and the Van der Pol oscillator, although nonlinear, also share this property. However, for certain nonlinear systems, e.g., the Duffing oscillator written down above, this property does not hold. These systems show extreme sensitivity to ini-



tial conditions which means that if there are two initial conditions infinitesimally close to each other then the trajectories which evolve from these neighbouring initial conditions will diverge exponentially, although the trajectories remain bounded.

The extreme sensitivity to initial conditions is an aid in understanding irreversibility. The time reversal invariance that we talked about earlier meant that if at an instant the velocities of all the particles in our system are reversed then the system will retrace its earlier path. This is quite correct if our system is isolated. In practice there can be no such thing as an isolated system – there will always be some interaction, however weak. Our system of gas molecules in the cylinder will feel the gravitational interactions of a distant object even if it is placed in vacuum. Reversing the velocities of the gas molecules in our cylinder does not produce a time reversed initial condition for the entire system which includes even the most distant particle in the universe. The infinite sensitivity to initial conditions will now prevent our gas molecules from retracing their paths exactly and the notion of invariance under time reversal loses significance.

The extreme sensitivity to initial conditions can also lead to a property of the trajectories which is known as ergodicity. The dynamical evolution of our system (the  $6N$  coordinates and momenta of our  $N$  gas molecules) can be pictured in the phase space. As explained before the state of our system at any instant is a point in phase space and as time goes on the evolution of our system can be seen as a trajectory in the phase space. The ergodic hypothesis says that in its motion through phase space, the point representing our system spends in each region a fraction of time proportional to the volume of the region. While the sensitivity to initial conditions is not a requirement of ergodicity, it certainly is a useful step in establishing ergodicity in cases where it can be established. Assuming ergodicity we can now go

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back to (18). The multiplicity of the final state in our time evolution is greater than the multiplicity of the initial state by a huge factor, which means the volume of the phase space associated with the final state is enormously larger than that associated with the initial state. The ergodic hypothesis now asserts that the fraction of time the phase trajectory spends in the final state is far greater than the fraction of time it spends in the initial state. Consequently, the time required to return to the initial state would be unimaginably big and will go to infinity in the thermodynamic limit of  $N \rightarrow \infty$ . The system consequently shows irreversibility. The key point is that we always start from a very improbable initial state. We thus have the central result that the macroscopic time evolution of our system is always such that it will settle in the state of maximum entropy.

We now show a standard result as an application of the statement that the total entropy of the universe can never decrease. It is useful to talk about a device known as an engine which works in cycles by absorbing heat from a reservoir and converting it into useful mechanical work. How efficient can this engine be? Obviously, it is not possible to convert the entire heat energy that is drawn from the reservoir into work because in that case the entropy of the reservoir would decrease and no change in entropy occurs elsewhere so that entropy of the universe would decrease. The process becomes feasible if not all the heat is converted into work, but instead a part of it is dumped into a reservoir at a lower temperature. If  $Q_1$  is the amount of heat drawn from the high temperature reservoir (temperature  $T_H$ ) and  $Q_2$  is the amount of heat dumped into the low temperature reservoir (temperature  $T_L$ ), then the change in entropy  $\Delta S = \frac{Q_2}{T_L} - \frac{Q_1}{T_H} = \frac{Q_1 - W}{T_L} - \frac{Q_1}{T_H} = Q_1 \left( \frac{1}{T_L} - \frac{1}{T_H} \right) - \frac{W}{T_L}$ , where  $W$  is the amount of work done so that  $Q_2 = Q_1 - W$ . Now since  $\Delta S \geq 0$ , we see that  $\frac{W}{Q_1} \leq 1 - \frac{T_L}{T_H}$ , which shows that the efficiency of the engine which is obviously  $\frac{W}{Q_1}$  is

limited by the quantity  $1 - \frac{T_L}{T_H}$ .

## Applications

We end by pointing out the generality and usefulness of a relation such as (10) in two widely different instances. The first is in information theory where an important concept is the information content of a message [4]. A key role is played by the length of the message, which tells us how hard the message is to guess. In determining the length of the message the alphabet is also important. Let us imagine the alphabet is composed of four letters A, T, G and C. If we are to use 0 and 1 to write out the message then we must ascribe a pair of numbers to each symbol e.g. 00 for A, 01 for T, 10 for G and 11 for C. The message which is a string of A, T, G and C in a definite order becomes a string of 0 and 1 in the corresponding order. The allowed number of messages are the various permutations of the digits. Since we are using 1 and 0 as the elements, for messages of 100 bits, the number of allowed messages will be  $2^{100}$ . The information content will be determined by the number of allowed messages and an information theoretic entropy (I) can be defined by a relation similar to (10), namely

$$I = K \log (\text{total number of allowed messages}).$$

The connection between entropy and information theory is based on the similarity of the above relation with (10).

The final example involves the curious astronomical objects called black holes [5]. These are generally massive objects, with a gravitational field so large that the escape velocity exceeds the velocity of light. Consequently, no signal comes out of this object. On the other hand, any object approaching a black hole is sucked into the hole and once it crosses the event horizon of the hole (an imaginary boundary around the star, beyond which, on the star-side time and space interchange character) is

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## Suggested Reading

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- [2] H B Callen, *Thermodynamics*, Wiley, NY, 2nd Ed., 1985.
- [3] R Baierlein, *Thermal Physics*, Cambridge University Press, 1999.
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- [5] S W Hawking in *Scientific American*, special issue, 'Cosmology + 1', 1977.

lost to the outside observer forever. Black holes can rotate and they can have charge. The properties of a black hole are determined completely by its mass, angular momentum and charge. It does not matter what kind of matter went into the formation of the black hole or how it was formed. This seems to be an ideal candidate for entropy to decrease. The cylinder of gas that we have been talking about could be taken to the vicinity of a black hole and the gravitational pull of the hole will eventually suck it in and the entropy contained in the cylinder would be lost for ever. If we assume that no entropy is defined for the black hole, then the total entropy of the universe is reduced in the process.

The way out was found on the basis of the observation that in any process the horizon area of the black hole never decreases. This led to the bold proposal that there should be a thermodynamic entropy, associated with the black hole, proportional to the horizon area. The total entropy (i.e. the black hole + the rest of the universe) can now increase. Since a black hole can be produced in a variety of different ways (it does not matter what it is made up of or how it was made so far as its thermodynamic characteristics go) it is again possible to write a relation similar to (10):

Entropy of the black hole =  $k \ln(\text{number of possible histories of the hole})$ .

Once quantum mechanics is taken into account, black holes do indeed turn out to have an entropy and a temperature, and these speculations are shown to have a firm basis [5]. Quite unexpectedly then, the black hole falls into the framework of statistical mechanics and thermodynamics.

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