Forever Undecided : A Puzzle Guide to Gödel

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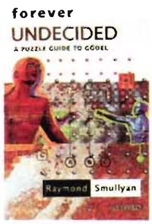

Forever Undecided: A Puzzle Guide to Gödel
by Raymond Smullyan
Oxford University Press
1987, pp.258

Welcome to the island of knights and knaves. There are three rules to life on this island: (1) The knights always speak the truth. (2) The knaves always make false statements. (3) Every native inhabitant of the island is a knight or a knave. Making sense of what people say on this island is a fascinating challenge for any visitor; if the visitor happens to be anyone interested in logic, there is no dearth of fun, and she can also discover beautiful theorems in logic.

Wandering around the island, you meet a native who says "you will never know that I am a knight." Startled, you start reasoning: "Suppose he is a knave. Then his statement is false, therefore at some time I *will* know that he is a knight, but I can't know that unless he really is one. Thus if he is a knave, then he must be a knight, a contradiction. Therefore he must be a knight."

You are happy, but your happiness is shortlived. You go on "Now I know that he is a knight, although he said I never would. Hence his statement is false, so he must be a knave." Paradox! you are in trouble now – is this a genuine paradox, or are you missing something?

This puzzle is almost the starting point for a whole range of puzzles related to 'introspective reasoners' in Smullyan's book. Note that it is different from a puzzle of the kind that asks: "What single question would you ask a native of the island to determine whether he is a knight or a knave?" The difference is that the latter relates to *classical propositional logic*, whereas the former is one in *propositional modal logic*, where we not only reason about what is true or false, but also about whether something is known or not.

Replace 'know' by 'believe' in the native's utterances and you begin to see the range of possibilities. Now the matter relates to whether the reasoner believes that the rules of the island hold, whether she believes that her beliefs are accurate, whether her beliefs are *closed* under logical consequences, etc. These possibilities give Smullyan room for a whole range of puzzles: we have reasoners who are peculiar (they may believe a proposition p without believing that they believe p), stable (if they believe that they believe p then they believe p as well) etc. You can imagine what fun is in store.

But then, as always, reading Smullyan is not only fun, but also a great opportunity to learn logic. Consider the island as an arena of mathematical statements – each statement is true or false (and not both). Rather than belief, consider *provability* of a proposition p.

Now it is easy to see that the discussion is about mathematical proof systems. Accurate beliefs above relate to soundness of the system: it derives only true statements. Also proving p may be possible without proving that such a proof exists, within the system – that is what peculiarity is about. Working out these puzzles, we collect a whole series of facts about proof systems, eventually leading to an understanding of Gödel's second incompleteness theorem, and from there on to Löb's theorem, the Kriplee-de Jough–Samlin theorem, and so on.

In 1931, Gödel proved his celebrated incompleteness theorems on the limitations of formal proof systems. The first theorem showed that in any system that contains arithmetic, there is a proposition p such that neither p nor its negation are provable in the system. His second theorem asserted that if the system is consistent, then this consistency cannot be proved in the system. The proofs of these theorems are tricky and Smullyan's book presents the proof through puzzles in such a way that any motivated school kid can understand.

What is remarkable is that the book goes further than Gödel's theorems, well into contemporary research territory. Modal logics of provability are of great current interest in mathematical logic and theoretical computer science, and I can well believe that readers of this book would be capable of launching into these research problems. Two or three (out of a total of thirty) may seem a bit too technical but that's a small price to pay for so much fun and learning.

And, if you care little for logic and research but just want challenging brainteasers, that's fine, just tuck in!

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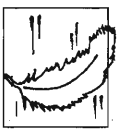

Then it was found that the rules of motions of particles were incorrect. The mechanical rules 'inertia' and 'forces' are wrong – Newton's laws are wrong – in the world of atoms. Instead, it was discovered that things on a small scale behave nothing like things on a larger scale.

Richard P Feynman
The Feynman Lectures on Physics
Addison Wesley Publishing Company
Reding, MA, 1983, p.2.6