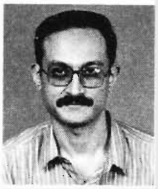
Symmetry in the World of Man and Nature

2. Frieze Groups

Shailesh A Shirali



Shailesh Shirali has been at the Rishi Valley School (Krishnamurti Foundation of India), Rishi Valley, Andhra Pradesh, for more than ten years and is currently the Principal. He has been involved in the Mathematical Olympiad Programme since 1988.

In Part 1 of this article[1], we had introduced the idea of symmetry as a mapping that maps a given object onto itself, and we studied an important class of mappings, the *isometries* – the maps that leave distances unchanged. We showed that every isometry is either the identity, a rotation, a reflection, a translation or a glide reflection. In this part, we consider some further properties of isometries and make some remarks on the Erlangen programme of Felix Klein; then we classify the so-called frieze groups.

1. Further Properties of Isometries

We now show an extremely important linearity property: *an isometry that fixes* 0 *is linear.* By this we mean that if f is an isometry with $f(0) = 0$, then for all $x, y \in \mathbf{R}^2$, we have $f(x + y) = f(x) + f(y)$, and if λ is an scalar quantity, then $f(\lambda x) = \lambda f(x)$. More generally, if λ, μ are scalars and $x, y \in \mathbf{R}^2$, then

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y).$$

We offer a geometric argument to prove the claims. Let O denote the origin and let A, B be points distinct from O; let $P = f(A), Q = f(B)$. Let C be determined by the condition that $OACB$ is a parallelogram, and let $R = f(C)$. Since f is an isometry, $OPRQ$ is congruent to $OACB$, so $OPRQ$ too is a parallelogram and it follows that $f(x + y) = f(x) + f(y)$. The proof that $f(kx) = kf(x)$ follows similarly, and the more general statement follows by combining the two statements.

A much stronger statement can be made: let f be any

[1] Part 1. Classification of Isometries, *Resonance*, Vol.6, No.5, pp.29-38, 2001.

A finite group of isometries of the plane is either cyclic or dihedral.

isometry (not necessarily) with a fixed point); let $a_1, a_2,$ $, a_n$ be points in $\mathbf{R}^2$, and let $\lambda_1, \lambda_2,$ $, \lambda_n$ be scalar quantities whose sum is 1. Then:

$$f(\lambda_1 a_1 + \lambda_2 a_2 + \quad + \lambda_n a_n) =$$

$$\lambda_1 f(a_1) + \lambda_2 f(a_2) + \quad + \lambda_n f(a_n).$$

The proof is left to the reader.

We shall use these remarks to prove the following theorem about the group $\mathcal{I}$ of isometries.

Theorem 1 *Let $\mathcal{G}$ be a finite subgroup of $\mathcal{I}$. Then there is a point in $\mathbf{R}^2$ which is fixed by every element in $\mathcal{G}$.*

Proof We first note that a finite subgroup of $\mathcal{I}$ cannot contain translations or glide reflections, as these have infinite order; it must consist entirely of reflections and rotations. Let a be any point in $\mathbf{R}^2$ and let S_a be defined by

$$S_a = \{f(a) : f \in \mathcal{G}\}.$$

This set is the *orbit* of a under $\mathcal{G}$; it is finite because $\mathcal{G}$ is finite. Note that each element of $\mathcal{G}$ is a symmetry for S_a, because each $g \in \mathcal{G}$ merely permutes the elements of S_a (i.e., $g(S_a) = S_a$). Let S_a have n distinct points, and let c be the centroid of S_a. Then $c = (\sum_{x \in S_a} x)/n$, and

$$f(c) = f\left(\frac{\sum x}{n}\right) = \sum \frac{f(x)}{n} = \frac{\sum f(x)}{n}.$$

(The summations are over S_a, e.g., $\sum f(x)$ means $\sum_{x \in S_a} f(x)$.) Since $f(S_a) = S_a$, the equality $\sum f(x) = \sum x$ holds, so $f(c) = c$; that is, c is fixed by f. As per the earlier remark, we see that each rotation in $\mathcal{G}$ is centered at c, and the axes of the reflections in $\mathcal{G}$ pass through c.

Theorem 2 *A finite subgroup of $\mathcal{I}$ is either cyclic or dihedral.*

Proof Let $\mathcal{G}$ be a finite subgroup of $\mathcal{I}$, and let c be a point fixed by all $g \in \mathcal{G}$. Consider the subgroup $\mathcal{G}_d$ of direct isometries in $\mathcal{G}$; it consists entirely of rotations centered at c. If ρ denotes the rotation in $\mathcal{G}$ with least possible (non-zero) angle of rotation, then $\mathcal{G}_d$ must be generated by ρ, so $\mathcal{G}_d$ is cyclic, say with n elements. If $\mathcal{G}$ consists only of direct isometries then $\mathcal{G}$ is isomorphic to the cyclic group Z_n, and we are through. If not, let σ be an indirect isometry in $\mathcal{G}$; then the products of σ with the elements of $\mathcal{G}_d$ are all indirect isometries and are distinct from one another, so the number of indirect isometries in $\mathcal{G}$ is at least n. On the other hand, the products of σ with the indirect isometries in $\mathcal{G}$ are all direct isometries and are distinct from one another, so the number of indirect isometries in $\mathcal{G}$ is at most n. It follows that the number of indirect isometries is exactly n. These consist of reflections in axes passing through c, and it is easy to see that in this case $\mathcal{G}$ is isomorphic to the dihedral group D_n. □

Leonardo da Vinci, made a detailed study of buildings with rotational symmetry (e.g., cathedrals), and worked out ways of attaching new structures to the buildings without destroying their symmetry.

Theorem 2 is sometimes attributed to Leonardo da Vinci, who made a detailed study of buildings with rotational symmetry (e.g., cathedrals), and worked out ways of attaching new structures to the buildings without destroying their symmetry; see Coxeter ([1]). (However, some authors have suggested that too much is being read into da Vinci's work.) Coxeter writes: "The prevalent groups in architecture have always been D_1 and D_2. But the pyramids of Egypt exhibit the group D_4, and ... in modern times the Pentagon building in Washington has the symmetry group D_5, and the Bahai temple near Chicago has D_9. ...The symmetry group of a snowflake is usually D_6 but occasionally only D_3. ...If you cut an apple the way most people cut an orange,[2] the core is seen to have the symmetry group D_5"

[2] The 'orange' referred to is known to us in India as 'sweet lime'.

2. Klein's Erlangen Programme

In 1872, at the University of Erlangen, Felix Klein delivered a by-now very famous lecture, on a new way of looking at geometry; it has now become known as the

The pyramids of Egypt have the symmetry group D_4; the Pentagon building in Washington has the symmetry group D_5.

Erlangen programme. His essential idea is that what distinguishes one geometry from another is the group of transformations under which its propositions remain unchanged. In the case of Euclidean geometry, the geometry with which we are most familiar, the propositions remain unchanged under the effect of isometries and also under the effect of scale change. Thus, a right angled triangle continues to be right angled after a change of scale, so Pythagoras's theorem remains valid. So the group of transformations that corresponds to Euclidean geometry includes not only isometries but enlargments too.

If we enlarge the transformations group, then different geometries result. Thus, we may introduce the *shear* operation (e.g. the mapping $(x, y) \mapsto (x + y, y)$; this is a shear parallel to the x-axis). Under this mapping neither right angles nor distances are preserved, but parallelism is preserved (in other words, the images of parallel lines are parallel too). Remarkably, in this geometry any two triangles are congruent to one another, as are any two parallelograms! The use of this group allows for simple proofs of certain theorems (e.g.,"the medians of a triangle are concurrent" or the famous nine-point circle theorem).

If the group is enlarged further to include *projections*, then we advance to projective geometry. Now, any two conics are congruent to one another! Several elegant theorems about conics get proved almost as corollaries; e.g Pascal's famous hexagon theorem. The shortest known proof of Morley's theorem (proved in 1899 or so) uses projective geometry.

We could now throw caution to the winds and introduce continuous movements into the picture. In the resulting geometry, straight lines may not get mapped onto straight lines, but closed curves do get mapped onto closed curves. The study of this geometry is essentially topology.

3. The Frieze Groups

The common feature of frieze patterns is translational symmetry.

In this section we plan to make a study of border patterns, also known as *frieze patterns*, of the types seen in temple friezes, *saree* borders, grill fencing around a garden, trellis work on a balcony, and so on. Their common feature is translational symmetry: in each case, the pattern repeats endlessly in one direction and so the group of symmetries of the pattern contains a non-zero translation; the subgroup of translational symmetries is infinite cyclic (isomorphic to the additive group of the integers). Examples from history are easy to list, and nice examples may be found in many ancient palaces and temples. In the frieze patterns shown below:

$$\clubsuit\ \clubsuit\ \clubsuit\ \clubsuit\ \clubsuit\ \clubsuit\ \clubsuit\ \clubsuit$$

$$\Uparrow\quad \Uparrow\quad \Uparrow\quad \Uparrow\quad \Uparrow\quad \Uparrow\quad \Uparrow$$

the symbols '$\clubsuit$' and '$\Uparrow$' are the *motifs* for the patterns. Note that though the motifs are different, their underlying symmetry is the same, as in the manner in which they have been stacked in line one after another. As a result, the symmetry groups for the two frieze patterns are the same. The frieze pattern

$$\Rightarrow\quad \Rightarrow\quad \Rightarrow\quad \Rightarrow\quad \Rightarrow\quad \Rightarrow\quad \Rightarrow\quad \Rightarrow$$

has another kind of motif and a different symmetry group. The line of footprints displayed in Part 1 also provides a nice example of a frieze pattern (note that it must be regarded as infinite in extent) and its symmetry group contains a glide. The same group is exhibited by the strip.

b p b p b p b p b p ..

We see thus that it is possible for the symmetry groups of two frieze patterns to be the same, even though the patterns differ in appearance. It turns out that if we put aside the details of the motif, retaining only the information that pertains to its symmetries, then there are just seven types of symmetry that a frieze pattern can have. We plan in this section to enumerate these symmetries.

We formally define a *frieze group* as follows. Let ℓ denote a fixed line in the plane $\mathbf{R}^2$, and let S_ℓ denote its group of symmetries (it has the following isometries: translations of all magnitudes, parallel to ℓ; reflections in all lines perpendicular to ℓ; half-turns centered at all points on ℓ; and composites of these mappings, e.g. glides). A *frieze group $\mathcal{F}$ with center ℓ* is a discrete subgroup of S_ℓ that contains a non-zero translation. (A 'discrete' subgroup in this context is one that does not contain translations by infinitely small amounts; or, in the 2-D case, rotations by infinitely small angles.)

Let τ denote a fixed (non-zero) translation parallel to ℓ; we take its length to define the unit of length. We shall now enumerate the frieze groups $\mathcal{F}$, centered on ℓ, whose subgroup of translations is the infinite cyclic group $\langle\tau\rangle$ generated by τ. (Note that this means that $\mathcal{F}$ has no translations with non-integer length). We shall show that *there are just seven such groups*. In other words, up to conjugation there are just seven frieze groups.

Notation For the analysis, let ℓ be the x-axis, let τ be the mapping that takes (x, y) to $(x+1, y)$, let α and β denote reflections in the x-axis and y-axis, respectively:

$$(x,y) \xrightarrow{\alpha} (x,-y), \qquad (x,y) \xrightarrow{\beta} (-x,y),$$

and let γ denote the glide reflection given by

$$(x,y) \xrightarrow{\gamma} \left(x+\frac{1}{2}, -y\right)$$

Note that $\alpha^2 = \beta^2 = \iota$, the identity map, and $\gamma^2 = \tau$.

Our first claim is that the frieze groups with center ℓ and subgroup of translations $\langle\tau\rangle$ can be generated by α, β, γ; i.e., all such frieze groups $\mathcal{F}$ are subgroups of $\langle\alpha, \beta, \gamma\rangle$.

There are just seven types of symmetry that a frieze pattern can have.

Suggested Reading

[1] H S M Coxeter, *Introduction to Geometry*, Wiley, 1961.
[2] Michael Artin, *Algebra*, Eastern Economy Edition.
[3] E George Martin, *Transformation Geometry: An Introduction to Symmetry*, Springer Verlag, 1982.
[4] L Tarasov, *This Amazingly Symmetrical World*, Mir Publishers, 1986.
[5] Elmer Rees, *Notes on Geometry*, Springer Verlag, 1985.
[6] Hermann Weyl, *Symmetry*, Princeton University Press, 1952.

Table 1.

$\mathcal{F}$	Frieze Pattern with Symmetry Group $\mathcal{F}$
$\langle\gamma^2\rangle$	... ℜ ℜ ℜ ℜ ℜ ℜ ℜ ℜ ...
$\langle\alpha,\gamma^2\rangle$	... ⇒ ⇒ ⇒ ⇒ ⇒. ...
$\langle\beta,\gamma^2\rangle$	... ∀ ∀ ∀ ∀ ∀ ∀ ∀ ∀ ∀ ∀ ...
$\langle\alpha\circ\beta,\gamma^2\rangle$	... ⇌⇌⇌⇌⇌⇌⇌ ...
$\langle\alpha,\beta,\gamma^2\rangle$	... ◇ ◇ ◇ ◇ ◇ ◇ ◇ ◇ ...

For proof, note that a reflection ($\neq \alpha, \beta$) contained in $\mathcal{F}$ must be in an axis perpendicular to ℓ. Let δ denote any such reflection. Then $f = \delta \circ \beta$ is a translation parallel to ℓ, so the displacement produced by f is of integral length (because, by assumption, no translation in $\mathcal{F}$ has a length of less than one unit). But in this case δ can be generated by β and γ^2. Next, note that $\alpha \circ \beta$ is a half-turn. If δ is any half-turn in $\mathcal{F}$, distinct from $\alpha \circ \beta$, then $\delta \circ \alpha \circ \beta$ is a translation, and as it is of integer length, it is an iterate of γ^2; therefore δ can be generated by $\alpha \circ \beta$ and γ^2. We argue similarly in the case of glide reflections (details left to the reader).

The second claim is that α and γ cannot occur together in a subgroup of $\mathcal{F}$. This holds because $\alpha \circ \gamma$ is a translation, namely $(x, y) \mapsto (x + 1/2, y)$, with length < 1.

With these observations, the subgroups can be enumerated. Suppose that $\mathcal{F}$ contains no glide reflection; i.e., $\mathcal{F}$ is a subgroup $\langle\alpha, \beta, \gamma^2\rangle$. We find that all the possible combinations can occur, and we enumerate them in *Table* 1, citing in each case a frieze pattern with a matching symmetry group.

Table 2.

$\langle\gamma\rangle$	$\langle\beta,\gamma\rangle$	$\mathcal{F}$
...	...	Frieze pattern with symmetry group $\mathcal{F}$
↗	⇑	
↘	⇓	
↗	⇑	
↘	⇓	
↗	⇑	
↘	⇓	
↗	⇑	
↘	⇓	
...	...	

Next, we consider frieze groups that do possess the glide γ. We have already noted that $\mathcal{F}$ cannot contain both α and γ, so there are just two possible combinations, $\langle\gamma\rangle$ and $\langle\beta, \gamma\rangle$. Both possibilities can occur, and we list them in *Table* 2.

It follows that up to conjugation there are precisely seven frieze groups.

Address for Correspondence
Shailesh A Shirali
Rishi Valley School
Chittoor District
Rishi Valley 517 352
Andhra Pradesh, India.