

# Flood Wave Propagation

## The Saint Venant Equations

*P P Mujumdar*



P P Mujumdar is with the Department of Civil Engineering, Indian Institute of Science, Bangalore. His research interests are mainly in the areas of water resources systems and stochastic hydrology and his areas of industrial consultancy include urban flooding, operations of large scale water resources systems and impact of engineering systems on the water environment.

**In this article, we will discuss the specific problem of how a flood wave propagates along a channel – a stream, a river or any open channel, and present the theoretical framework available for forecasting the propagation of the flood wave.**

### Introduction

Among all natural disasters, floods are the most frequently occurring phenomena that affect a large section of population all over the world, every year. Throughout the last century, flooding has been one of the most devastating disasters both in terms of property damage and human casualties. The primary effects of floods, caused by direct contact with the floods include, apart from the loss of human life, damages caused by water entering built structures, extensive erosion caused due to the high velocity of flood waters and loss of crop and live stock due to flooding of farmlands. Flood water can also concentrate garbage, debris and toxic pollutants that can cause the secondary effects of health hazards.

When a large body of water rises and overflows onto a normally dry land, we call it a 'flood'. To a hydrologist, a flood is simply a discharge rate in a stream or a river that exceeds an 'acceptable' threshold value, and is often an expression of the variability of rainfall in the river catchment. The intensity and duration of the rainfall that can cause a given magnitude (discharge rate) of a flood vary from one hydrologic region to another, and in fact, depends on the aerial extent of interest. Flash floods, the floods that occur with little warning time are not only caused by excessive rainfall of high intensity but also by sudden release of high discharges of water from dams and by breaches or structural failures of dams and levees, gushing out large volumes of water in a short time.

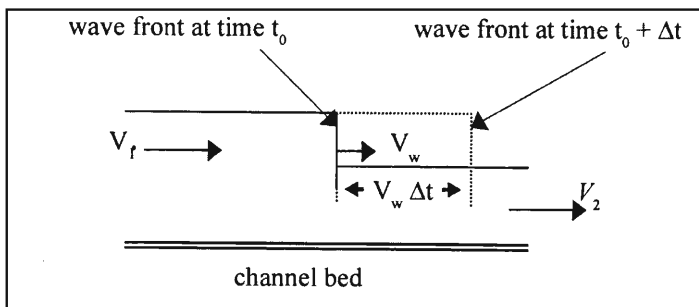
A flood travels along a river reach as a wave, with velocity and depth continuously changing with time and distance. While it is difficult to forecast with accuracy the time of occurrence and magnitude of floods, it is possible to predict fairly accurately the movement of the flood wave along a river, once it is known that a flood wave is generated at some upstream location in the river. Such a forecast is of immense practical utility, as it can be used in early flood warning systems. The topic discussed in the article is a small section of the broad area of study called the 'flood hydrology'.

### Flood Routing

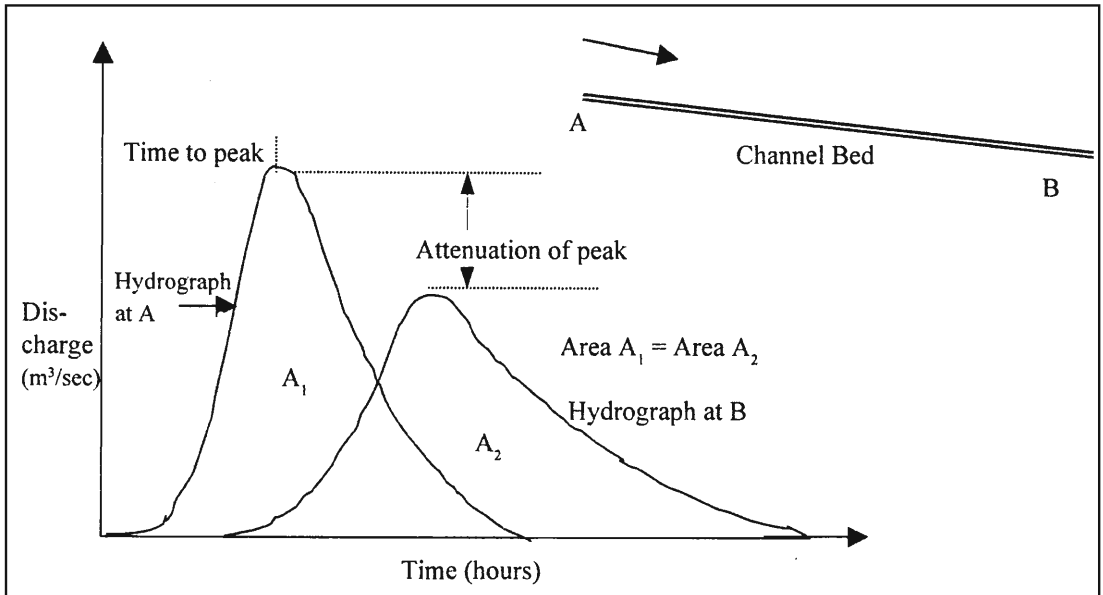
Imagine a flood wave traversing along a straight reach of a stream, which initially has uniform flow conditions. As the flood wave crosses a section, the velocity (or discharge) and the depth of flow at that section change. Our interest will be to study these changes in depth and discharge, with time and along the length of the river. In *Figure 1*, a monoclinal rising wave (a uniformly progressing translatory wave of stable form) is shown travelling along a small section of a stream. For such a wave, the successive positions of the wave front at different times are parallel and the wave configuration travels downstream at a constant velocity. The wave front is moving at a velocity, or celerity (velocity of the wave front), of  $V_w$ , which is greater than the mean velocities,  $V_1$  and  $V_2$  in the uniform flow regions upstream and downstream of the wave.

This is an instantaneous picture of a simplistic wave as seen at a section. For such a simple case, it is possible to express the

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**Figure 1.** Monoclinal rising wave in a channel.



**Figure 2. Flood hydrographs.**

celerity  $V_w$  in terms of the velocities  $V_1$  and  $V_2$ , given the cross-sectional areas at the two sections, upstream and downstream of the wave front. In real situations, however, the depth and discharge of the wave front itself change with time. A flood wave is, therefore, represented by a hydrograph (graph showing variation of discharge or flow depth with time), as shown in *Figure 2*, and the propagation of such a wave is quite different from that of a monoclinical wave. In practical applications, we will be interested in estimating the discharge and depth of flow at a given location along the stream at a specified time, given the flood hydrograph at an upstream section. Computation of the movement of flood wave along a channel with time and space is called 'flood routing'.

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### Saint Venant Equations

The flood flow is unsteady – as the flow properties (depth and velocity) change with time – and is gradually varied, because such change with time is gradual. To derive the governing equations for such a wave movement, we use the principles of continuity (conservation of mass) and momentum (essentially,

Newton's second law of motion) for the one dimensional unsteady open channel flow (see *Box 1*).

In differential form, the governing equations are written as,

Continuity:

$$\partial Q/\partial x + \partial A/\partial t - q = 0. \quad (1)$$

**Box 1. The Saint Venant Equations**

Consider an elemental control volume of length  $dx$  in a channel (*Figure 3*). The inflow to the control volume is the sum of the flow  $Q$  entering the control volume at the upstream end of the channel and the lateral inflow  $q$  entering the control volume as a distributed flow along the channel. The dimensions of  $q$  are those of flow per unit length, so the rate of lateral flow is  $qdx$ . The mass inflow rate to the control volume is,  $\rho(Q+qdx)$ . The mass outflow rate from the control volume is  $\rho(Q + \partial Q/\partial x dx)$  where  $\partial Q/\partial x$  is the rate of change of channel flow with distance. The volume of the channel element is  $A dx$ , where  $A$  is the average cross-sectional area. The rate of change of mass stored within the control volume is thus,  $\partial(\rho A dx)/\partial t$ . The continuity of mass, viz., outflow + change in storage = inflow, is written for the control volume, as

$$-\rho(Q+qdx) + \rho(Q + \partial Q/\partial x dx) + \partial(\rho A dx)/\partial t = 0.$$

For water, the density  $r$  may be assumed constant. Dividing throughout by  $rdx$ , we write the continuity equation as,

$$\partial Q/\partial x + \partial A/\partial t - q = 0.$$

The momentum equation is derived from Newton's second law, viz., rate of change of momentum = net force. As the derivation is rather lengthy, only a broad outline is given here. The forces considered for the control volume are  $F_g$ , the gravity force,  $F_f$ , the friction force,  $F_c$ , the contraction/expansion force,  $F_w$ , the wind shear force on the water surface and  $F_p$ , the unbalanced pressure force (net force due to  $F_{pl}$ ,  $F_{pb}$ ,  $F_{pr}$  in *Figure 3*). The net outflow of momentum from the control volume is computed from the mass inflow,  $\rho(Q+qdx)$ , and the mass outflow,  $\rho(Q + \partial Q/\partial x dx)$ , considering the average velocity,  $V$ , in the control volume and the  $x$ -component  $v_x$  of the velocity of the lateral flow,  $q$ . The volume of the elemental channel considered is  $A dx$ , its momentum is  $\rho A dx V$ , or  $\rho Q dx$ . The rate of change of momentum storage in the control volume, thus is,  $(\rho \partial Q/\partial t dx)$ . A momentum coefficient  $\beta$  is introduced to account for non-uniform distribution of velocity at a section in computing the momentum. The complete form of the momentum equation is written as,

$$\partial Q/\partial t + \partial(bQ^2/A)/\partial t + gA(\partial y/\partial x - S_0 + S_f + S_e) - \beta qv_x + W_f B = 0,$$

where  $S_0$  is the bed slope,  $S_f$  is the energy slope,  $S_e$  is the eddy loss slope,  $W_f$  is the wind shear factor, and  $B$  is the average top-width of water surface in the control volume. The momentum equation (2) in the text is a simplified form with  $\beta = 1$ , neglecting lateral flow, wind shear and eddy losses.



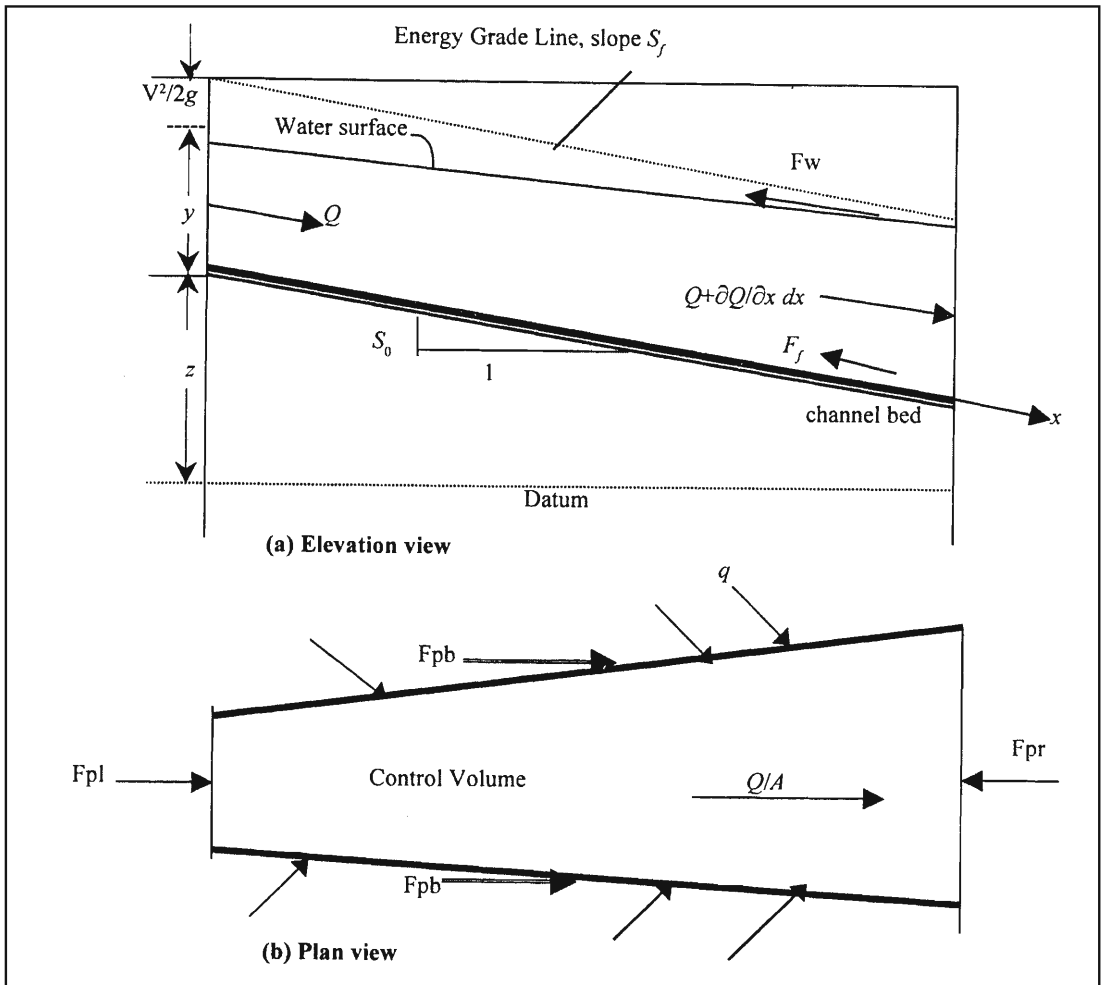
**Momentum:**

$$(1/A) \partial Q/\partial t + (1/A) \partial (Q^2/A)/\partial x + g \partial y/\partial x - g (S_0 - S_f) = 0 \quad (2)$$

Local acceleration term	Convective accelera- tion term	Pressure force term	Gravity force term	Friction force term
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In these equations,  $Q$  is the discharge ( $m^3/sec$ ),  $A$  is the area,  $q$  is the lateral flow per unit length of the channel ( $m^3/sec/m$ ),  $x$  is the distance along the channel,  $y$  is the depth of flow,  $g$  the acceleration due to gravity,  $S_0$ , the bed slope of the channel and  $S_f$  is the friction slope (see Figure 3). Equations (1) and (2) are together called the Saint Venant equations, in honor of Saint Venant (see

**Figure 3. Definition sketch for continuity and momentum equations.**



**Box 2. Saint Venant (1797 - 1886)**

Saint Venant was a French engineer whose major contributions were in the field of elasticity. He also wrote extensively on hydraulics, with particular attention to wave motion, pipe and open channel resistance, gradually varied flow and the flow of air. The following contributions are associated with his name: analysis of the sonic orifice, introduction of the term 'celerity' to distinguish the velocity of wave propagation with respect to the fluid from the velocity of the fluid itself and differentiation between rivers and torrents in terms of the critical slope for uniform flow. *The History of Hydraulics* by Rouse and Ince has this to say of Saint Venant: "...as is often the fate of those who write prodigiously in many fields, no one of his many publications on fluid motion stands above the others, and his net influence upon hydraulics may well have lain in the stimulating effect which these had as a whole upon his colleagues and successors".

**Box 2) who developed these equations in 1871. In the momentum equation, the local acceleration term describes the change in momentum due to change in velocity over time, the convective acceleration term describes the change in momentum due to change in velocity along the channel, the pressure force term denotes a force proportional to the change in water depth along the channel, the gravity force term denotes a force proportional to the bed slope, and the friction force term denotes a force proportional to the friction slope.**

It is not possible to solve (1) and (2) together analytically, except in some very simplified cases. Numerical solutions are possible, and are used in most practical applications. Depending on the accuracy desired, alternative flood routing equations are generated by using the complete continuity equation (except the lateral flow term, in some cases) while eliminating some terms of the momentum equation. Based on the terms retained in the momentum equation, the flood wave is called the kinematic wave, the diffusion wave and the dynamic wave, as shown:

$$(1/A) \partial Q / \partial t + (1/A) \partial (Q^2/A) / \partial x + g \partial y / \partial x - g(S_0 - S_f) = 0 \tag{3}$$

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Kinematic wave

Diffusion wave

Dynamic wave

The kinematic wave, is thus represented by,  $g(S_0 - S_f) = 0$ , or  $S_0$



$= S_f$ , the diffusion wave by,  $g \frac{\partial y}{\partial x} - g(S_0 - S_f) = 0$  resulting in  $\frac{\partial y}{\partial x} = (S_0 - S_f)$ , and the dynamic wave by the complete momentum equation (3).

In most practical applications (see *Box 3*), either the wave resulting from the simplest form of the momentum equation, viz., the kinematic wave, or from the complete momentum equation, viz., the dynamic wave is used. For the kinematic wave, the acceleration and pressure terms in the momentum equation are neglected, and hence the name kinematic, referring to the study of motion exclusive of the influence of mass and force. The remaining terms in the momentum equation represent the steady uniform flow. In other words, we consider the

### Box 3. Some Applications of the Saint Venant Equations

Numerical solutions of the Saint Venant equations are used to predict the flood arrival time and its magnitude (i.e., the flood hydrograph) at various locations along a river, once the flood hydrograph at an upstream location is known. The typical distances we are concerned here are in hundreds of kilometers, times of the order of 2-3 days with routing time steps of one hour, and the typical flood discharges may be of the order of thousands of cumecs ( $m^3/sec$ ). An accurate prediction of the travel of the flood wave during the unprecedented floods in the Yamuna in 1978, led to the timely evacuation of parts of Delhi.

The dynamic wave model is used to analyse the flash floods caused due to dam breaks. Downstream flood wave caused by the 1976 failure of the Teton dam in Idaho, US was re-constructed by the dynamic wave model. The flash floods had rendered 25,000 people homeless, apart from causing huge economic loss. The re-construction of the hydrograph downstream of the dam showed that the peak reservoir outflow was about 20 times greater than the largest recorded floods at that site.

Flood routing is used in designing the urban storm water drainage systems. Urbanisation modifies the hydrologic characteristics of a region. The hydrographs after urbanisation show a higher peak and a lower time-to-peak, for the same rainfall pattern. This is due to the quicker and higher runoff because of the increased impervious area. As impervious area increases, therefore, drainage of storm water becomes challenging. The kinematic wave model is used to route the flood hydrographs through the urban drainage system, to examine the adequacy of the system.

In engineering designs for flood control (e.g., flood control reservoirs), a design flood hydrograph is routed to ensure the safety of the design. The design flood, such as a 100 year flood (*Box 4*), is selected based on the perceived acceptable risks. The flood hydrograph used for routing is constructed based on the design duration, which depends on the hydrology of the region.



flow to be steady for momentum conservation but take into consideration the effects of unsteadiness through the continuity equation. Analytical solution is possible for the simple case of the kinematic wave, where the lateral flow is neglected and the wave celerity is constant. However, the backwater effects (propagation upstream, of the effects of change in depth or flow rate at a point) are not reproduced through a kinematic wave. Such backwater effects are accounted for in flood routing only through the local acceleration, convective acceleration and the pressure terms, all of which are neglected in the kinematic wave. For more accurate flood routing, numerical solutions of the complete dynamic equation are used.

### Numerical Solutions

With the availability of high speed computers, it is now possible to solve the complete dynamic wave equations through numerical methods (such as the finite difference methods). The numerical methods start with initial and boundary conditions. At time  $t=0$ , the uniform steady flow conditions are specified at all locations. These constitute the initial conditions. At distance  $x=0$ , the flood hydrograph is known. This, together with other flow conditions (such as free over-fall at a location, flows from other sources joining the channel, submergence at a junction, etc.) along the length of the channel define the boundary conditions. Solution of the Saint Venant equations gives the variation of discharge (and depth) with time, along the length of the water body (river, stream or a channel) which is used for real-time flood forecasting. Our ability for accurate real-time flood forecasting has significantly improved because of sophistication in the numerical techniques used to solve the dynamic wave equations. It must, however, be noted that the real world flood routing problems are quite complex because of several types of boundary conditions that need to be included in the routing models, and even with the present day computers, solution of the dynamic wave equations can take several hours of computer time.

#### Box 4. What is a 100 Year flood?

A 100-year flood is so big and rare that it will normally occur only once in every 100 years. That does not mean that a 100 year flood cannot occur immediately in the year after occurrence of a 100 year flood! Every year has exactly the same probability – one in 100 – of producing a 100 year flood, even in the area that experienced a 100 year flood in the year before.

### Suggested Reading

- [1] V T Chow, D R Maidment and L W Mays, *Applied Hydrology*, McGraw Hill, 1988.
- [2] K G Rangaraju, *Flow through Open Channels*, Tata-McGraw Hill, 1993.

*Address for Correspondence*  
 P P Mujumdar  
 Department of Civil Engineering  
 Indian Institute of Science  
 Bangalore 560012, India.  
 Email: pradeep@civil.iisc.ernet.in

