Symmetry in the World of Man and Nature

1. Classification of Isometries

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1. Introduction

Symmetry as an idea has an aspect of universality to it. In virtually every facet of human endeavour and natural phenomena, cutting across the boundaries of time and space, we find manifestations of symmetry. As Hermann Weyl writes in his wonderful book [1], a book that is certainly essential reading for anyone with an interest in the subject, "Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection" The all-embracing nature of the concept of symmetry is staggering: within its fold lie subjects as far removed from one another as particle physics, relativity, crystallography, rangoli patterns and Islamic art. Scientists are far from being the only ones to ponder about the concept, and William Blake's immortal peoms reflect man's age-old fascination with symmetry:

Tyger! Tyger! burning bright
In the forests of the night,
What immortal hand or eye
Dare frame thy fearful symmetry?

Those keen on the analysis of poetry will note that Blake himself chooses to break the symmetry in the end, with his use of the word 'symmetry'. Here is Wordsworth:

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the palm of your hand
And Eternity in an hour.

(What symmetry is being reflected in these lines?)

Some of the finest examples of symmetry come from nature itself: the striking axial symmetry of a butterfly's wings, the rotational symmetry of a flower ("Nature's gentlest children" in Weyl's words), the astonishing 3-dimensional symmetry of certain radiolaria and pollen grains, the spiral arrangement of seeds in a sunflower, the logarithmic spiral on a snail shell, the spiralling double helix of the DNA molecule, etc.

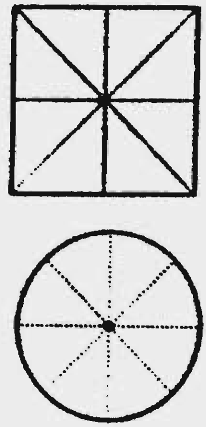
It has been said that 'symmetry is death'. The pun is suggestive – 'symmetry' and 'cemetery'(!); the purport presumably is that symmetry carries an association of stasis and lack of change, whereas life is ever changing, ever moving. But in fact some of the finest examples of symmetry come from nature itself: the striking axial symmetry of a butterfly's wings, the rotational symmetry of a flower ("Nature's gentlest children", in Weyl's words), the astonishing 3-dimensional symmetry of certain radiolaria and pollen grains, the spiral arrangement of seeds in a sunflower, the logarithmic spiral on a snail shell, the spiralling double helix of the DNA molecule, etc. It is one of the strange facts about the world that life has the urge as well as the capacity to create symmetric forms. Another curious thing is the phenomenon of left-right asymmetry in some organic molecules (for example, sugars) and the preference of living forms for one type of orientation. Why this should be so, and whether symmetry is inevitable in any form of life, is something the reader could reflect upon.

In mathematics too, the concept of symmetry has played a strongly unifying role. "The investigation of the symmetries of a given mathematical structure has always yielded the most powerful results..." wrote Emil Artin. The example that comes most immediately to mind is Felix Klein's work on the unification of geometry (the so-called *Erlangen programme* – see Part 2 in the forthcoming issue).

In this two-part expository article, we make a brief survey of the basic principles of symmetry. We discuss the different kinds of symmetry that an object can have, and note how the symmetries of an object form a group in a very natural manner. Following this, we classify the finite 2-dimensional symmetry groups. This is followed by an excursion into the study of repeating patterns: strip patterns (also known as frieze patterns or border patterns), wall-paper patterns and crystals. The exposition is largely self-contained, though we presuppose some knowledge of elementary group theory and linear algebra. The subject is a rich and fascinating one and only "extravagant incompetence on the author's part" (as Hardy might have put it, in his inimitable language) will fail to bring out its beauty.

2. The Concept of Symmetry

What exactly is symmetry? Everyone would agree that a square is symmetric in the line joining the midpoints of a pair of opposite sides and in each of its diagonals; likewise, that a circle is symmetric in any of its diameters (see *Figure* 1). A circle when reflected in any of its lines of symmetry falls back upon itself, as does a square. These considerations motivate the modern approach to symmetry. (Much of what is stated below has been done keeping in mind 2-dimensional space. This is only for the sake of simplicity; the same treatment works for 3-dimensional space.)



The square has essentially eight symmetries: reflections in the dotted lines shown, plus four rotations about its centre; these include the identity operation which corresponds to rotation by 0°. The circle has infinitely many symmetries: reflections in all its diagonals, and rotations about its centre by any angle whatever.

Figure 1. Symmetries of a square and a circle.

Let the underlying 2-dimensional space in which the objects under consideration are embedded be denoted by $\mathbf{R}^2$, and let the distance between points x and y in $\mathbf{R}^2$ be denoted by $d(x, y)$. Consider a mapping f of $\mathbf{R}^2$ into itself; define f to be an *isometry* if it leaves all distances unaltered, that is, if $d(f(x), f(y)) = d(x, y)$ for all x and y ('iso' means 'the same' and 'metric' has the connotation of distance). Further, define the isometry to be *direct* if it preserves orientation, and *indirect* if it causes a reversal of orientation. By this we mean the following: let an isometry f act on ΔABC, and let it take A to A', B to B' and C to C' Then f is *direct* if the direction of the cycle $A' \to B' \to C' \to A'$ is the same as that of the cycle $A \to B \to C \to A$; if not, it is *indirect*. (Alternatives to 'direct' and 'indirect' are *even* and *odd*.)

Isometries are familiar objects; examples are reflection in a line and rotation about a point, as also the translation $\mathbf{T}_a$ given by $\mathbf{T}_a(x) = x + a$, where a is any fixed ele-

Groups were first invented to analyze symmetries of certain algebraic structures called field extensions, and as symmetry is a common phenomenon in all sciences, it is still one of the main ways in which group theory is applied.

ment of $\mathbf{R}^2$. Note that reflection is an indirect isometry, whereas rotations and translations are direct isometries.

Symmetry Group of an Object

Notations: We shall denote points by uppercase letters and lines by lowercase letters. Rotations will be denoted by 'ρ' (with a subscript to show the centre of rotation) and sometimes by 'Rot'; reflections will be denoted by 'σ' (with a subscript to show the axis of reflection). Half-turns too will be denoted by 'σ'. So σ_P denotes a half-turn about the point P (i.e., a rotation through 180 degrees), and σ_l denotes reflection in the line l.

Let $\mathcal{I}$ denote the set of all isometries in $\mathbf{R}^2$. It is trivial to verify the following:

- The identity mapping ι, given by $\iota(x) = x$ for all x, belong to $\mathcal{I}$.
- If $f, g \in \mathcal{I}$, then the composition $f \circ g \in \mathcal{I}$.
- If $f \in \mathcal{I}$ then f possesses an inverse $f^{-1} \in \mathcal{I}$.

Indeed, $\mathcal{I}$ forms a group under functional composition, the *group of rigid motions* of $\mathbf{R}^2$, denoted (by abuse of notation) by the same symbol $\mathcal{I}$. (Note that a reflection is its own inverse; i.e., its order is 2.)

Exercises

1. Show that the set of all possible translations in $\mathbf{R}^2$ is a subgroup of $\mathcal{I}$.
2. Show that the set of all possible rotations (about all possible points) does not form a subgroup of $\mathcal{I}$, but that the set of all possible rotations together with the set of all possible translations does form a subgroup.

Let an object X be given, and consider the isometries f

in $\mathcal{I}$ that leave X fixed; that is, $f(X) = X$ (the points of X exchange places amongst themselves). If X were a square, then reflection in either diagonal would qualify for such a mapping, as would rotation about its center by 90 degrees. The set of all such isometries is called the *group of symmetries* of X and is denoted by the symbol $S(X)$; its elements are the *symmetries* of X Examples are easy to list: a butterfly has two symmetries, the identity map and a reflection; a 3-petalled flower has three symmetries, the identity map and rotations through 120 degrees and 240 degrees about its center; and so on. *Table* 1 displays the number of symmetries corresponding to a few familiar objects.

X	$\lvert S(X)\rvert$
Isoceles triangle	2
Parallelogram	2
Rectangle	4
Equilaterial triangle	6
Square	8
Circle	∞

Table 1. Orders of a few symmetry groups.

(Note that 'isosceles' means 'isosceles non-equilateral' 'rectangle' means 'non-square rectangle' and 'parallelogram' means 'non-rectangular 'parallelogram'.)

Exercises

3. List the eight symmetries of a square.
4. Show that a regular n sided polygon possesses $2n$ symmetries. (Its group of symmetries is the *dihedral group of order* $2n$, denoted by the symbol D_n. It contains n rotations and n reflections.)
5. Find a familiar man-made object in daily use whose symmetry group is isomorphic to Z_3. (Why is it Z_3 and not D_3?)

It is interesting to ask whether, given an arbitrary group G, we can find an object X for which $S(X)$ is isomorphic to G; this is equivalent to asking for a classification or enumeration of all the subgroups of $\mathcal{I}$. For instance, is there an object X for which $S(X) \cong Z_4$ (the cyclic group of order 4)? Yes indeed! A swastika (either one will do, the Nazi or the Hindu swastika) is such an object (see *Figure* 2). The corresponding 3-armed symbol has

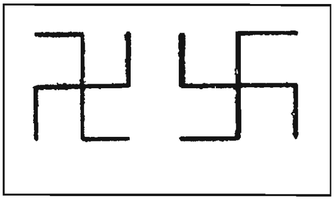


Figure 2. Nazi and Hindu swastikas.

a symmetry group isomorphic to Z_3 (the cyclic group of order 3). Obviously, the groups Z_n are all 'realizable' in the sense we have in mind.

Moving to three dimensions, we find a small complication. In two dimensions, all symmetries are realizable by genuine physical movements. For instance, reflection in a line can be physically achieved by rotation about the line by 180 degrees; this mirrors the actual situation exactly, including the orientation reversal caused by reflection. (In fact this is the only way of doing it. No amount of sliding about on a plane will ever achieve a reversal of orientation.) However in three dimensions, reflection in a plane cannot be physically realized. Thus, a left shoe cannot be transformed into a right shoe, no matter how you move it about – the only way to do so would be to make a quick dash to 4-dimensional space! (This brings attention to an unexpected hazard of journeying to higher dimensional spaces: a carelessly dropped shoe may reverse its 'parity', and we may be left with two left shoes or two right shoes. Astronauts, be forewarned!) From this point on, when we refer to 3-dimensional space we shall exclude from consideration all orientation-reversing isometries.

In three dimensions, reflection in a plane cannot be physically realized. Thus, a left shoe cannot be transformed into a right shoe, no matter how you move it about — the only way to do so would be to make a quick dash to 4-dimensional space!

Exercises

6. How many symmetries does a regular tetrahedron have? A regular octahedron?
7. Show that a cube has 24 symmetries. How many of these have order 2? Order 3? Order 4? Order 6? What physical movement corresponds to a symmetry of order 3?
8. Consider a 2-coloured football whose surface is a symmetrical mosaic of regular pentagons and hexagons, with the pentagons of one colour and the hexagons of another colour. How many symmetries does the football have?

9. What symmetries does an infinite helix have? (A spring offers a convenient model; see *Figure* 3.)

Figure 3. A helix.

10. The toy, 'Rubik cube', marketed in the early 1980's by the Hungarian architect Erno Rubik is a $3\times3\times3$ cube divided into three layers perpendicular to each of its three principal axes (see *Figure* 4). The internal structure of the cube is such that each layer can be rotated about its centre. Each of the outer layers has a different colour, so after a few such rotations the colours get hopelessly scrambled. The challenge is to restore it to its original pristine state. (The task is decidedly non-trivial!) The problem we pose here is to find the order of the group generated by the six basic movements.

    Warning: The number is extremely large, and the group has a very intricate structure! Perhaps this may account for the notorious difficulty of the puzzle.

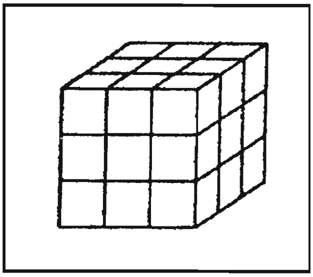


Figure 4. The Rubik cube.

Problem: How many configurations are possible for the Rubik cube? That is, what is the order of the group generated by the six basic motions of the cube (the rotations through 90° of each of the six faces)?

3. Isometries in Two Dimensions

We now proceed to classify the isometries in two dimensions. As mentioned earlier, the isometries in $\mathbf{R}^2$ include the translations, rotations and reflections. There is a fourth type of isometry, the *glide reflection*, which we define as follows. Let ℓ be a line in the plane, let σ_ℓ denote reflection in ℓ, let a be a non-zero vector parallel to ℓ and let $\mathbf{T}_a$ denote the translation map given by $\mathbf{T}_a(x) = x + a$. Then the product $g = \sigma_\ell \circ \mathbf{T}_a$ is referred to as a glide reflection. Note that $\sigma_\ell \circ \mathbf{T}_a = \mathbf{T}_a \circ \sigma\ell$. A glide has infinite order, and it reverses orientation. It is easy to exhibit a physical object that possesses glide symmetry: footprints on a beach! (See *Figure* 5.)

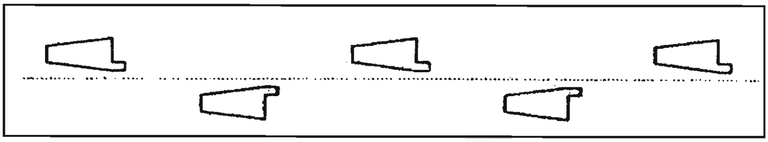


Figure 5. Schematic picture of footprints on a beach.

Theorem 1. *An isometry that fixes two distinct points fixes the entire line passing through them. An isometry that fixes three non-collinear points is the identity map.*

Proof. Let f be an isometry that fixes two points A, B. Then, any point P on the line AB is uniquely specified by the two distances AP, BP, and as distances are left unchanged, P too must be fixed by f. The second assertion is proved similarly.

Theorem 2. *An isometry that fixes two distinct points is either a reflection or the identity map.*

Proof. Let f be an isometry that fixes two points A, B; then, it fixes line AB, pointwise. Let C be a point not on AB. Since the distances AC, BC are fixed, $f(C)$ can be in one of just two possible locations. If $f(C) = C$ then f is the identity. If not, let g be the reflection in the line AB. Then, $g \circ f$ fixes A, B, C and hence is the identity. Thus, f is the inverse of g and so, $f = g$.

Theorem 3. *An isometry that fixes exactly one point is a product of two reflections. An isometry that fixes a point is a product of at most two reflections.*

Proof. Let f be an isometry that fixes exactly one point A. Let B be another point, and let $B' = f(B)$; then B, B' are equidistant from A, so A lies on line m which bisects BB' at right angles. Let g denote reflection in m. Then $g \circ f$ fixes A, B, so $g \circ f$ is either a reflection or the identity map. The latter leads to $f = g$, which cannot be, as f fixes exactly one point. Therefore $g \circ f$ is some reflection h. Thus, $f = g \circ h$. The second assertion follows as a corollary of the proof. □

Theorem 4. *Any isometry in 2-dimensional space can be expressed as a product of no more than three reflections.*

Proof We need only consider the case when the isometry f has no fixed points. Let P, Q be points with $Q = f(P)$.

Let m be the line bisecting PQ at right angles, and let g denote reflection in m. Then $g \circ f$ fixes P and so is a product of at most two reflections. Therefore f is a product of at most three reflections. □

Theorem 5. *Given two coplanar triangles congruent to one another, there exists a unique isometry mapping one triangle onto the other.*

The proof is left to the reader (Exercise 14).

Theorem 6.
(a) *The product of a translation and a reflection (in either order) is either a reflection or a glide.*
(b) *The same conclusion holds for the product of a rotation and a reflection (in either order).*

Proof. The simplest approach is via coordinatization. Let the axis of reflection be chosen to be the x-axis. Denote the reflection map by σ, and let the translation $\mathbf{T}$ be given by $\mathbf{T}(x, y) = (x + a, y + b)$; let $f = \sigma \circ \mathbf{T}$. Then f takes (x, y) to $(x + a, -y - b)$:

$$(x, y) \xrightarrow{T} (x + a, y + b) \xrightarrow{\sigma} (x + a, -y - b).$$

Define auxiliary maps α and β as follows: $\alpha(x, y) = (x + a, y), \beta(x, y) = (x, -b - y)$. Then α is a translation along the x-axis, while β is a reflection in the line $y = -b/2$. By computation, we see that $\sigma \circ \mathbf{T} = \alpha \circ \beta$. It follows that $\sigma \circ \mathbf{T}$ is a glide or a reflection (the latter case corresponds to $a = 0$; when $a \neq 0$, the axes of the maps $\mathbf{T}$ and σ are parallel, so $\sigma \circ \mathbf{T}$ is a glide by definition). The proof that $\mathbf{T} \circ \sigma$ is a reflection or a glide is handled similarly, and we leave the proof of part (b) to the reader. □

Theorem 7. *Any isometry in $\mathcal{I}$ is the identity, a translation, a rotation, a reflection or a glide reflection.*

Proof. As earlier, we need only consider the case when the isometry f has no fixed points; then by Theorem 4, f can be expressed as a product of three reflections. Since the product of two reflections is either the identity, a translation or a rotation (see Exercise 13), we can invoke Theorem 6. The result follows. □

Exercises

11. Let $\sigma_P, \sigma_Q, \sigma_R$ denote half-turns in three distinct points P, Q, R. Show that

$$\sigma_P \circ \sigma_Q \circ \sigma_R = \sigma_R \circ \sigma_Q \circ \sigma_P.$$

12. Let m be any line, let σ_m denote reflection in m, and let f be any isometry. Show that the isometry $f \circ \sigma_m \circ f^{-1}$ is the same as reflection in the line $f(m)$.

13. Let m, n be given lines. Show that the composite map $\sigma_m \circ \sigma_n$ is either the identity map (if m and n are the same), a translation (if $m \parallel n$), or a rotation (if m and n are non-parallel; in this case, the centre of rotation is the point $m \cap n$).

14. Prove Theorem 6.

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Suggested Reading

[1] Hermann Weyl, *Symmetry*, Princeton University Press, 1952.
[2] George E Martin *Transformation Geometry: An Introduction to Symmetry*, Springer Verlag.
[3] L Tarasov, *This Amazingly Symmetrical World*, Mir Publishers, 1986.
[4] D'Arcy Thomson, *Growth and Form*, Cambridge University Press, 1917.

Absence of evidence is
not evidence of absence.

Carl Sagan
The Dragons of Eden