Computer Based Modelling and Simulation

1. Modelling Deterministic Systems

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Simple models for deterministic systems are explained in this part. These models are useful in engineering, business, economics and sociology.

Introduction

Computer simulations are 'experiments' using computers to realize meaningful results. They are less costly, less time consuming and much safer than actual physical experiments. We shall discuss briefly the process of developing simulation procedures and point out certain guidelines with simple examples.

Simulations are done in two steps. The first step is to build reliable and robust models. Then simulations are performed as the second step with carefully chosen inputs and parameters. [Remember the old dictum of computer users: "Garbage in – Garbage Out"]. If the inputs and parameters are not appropriate, models and simulations will give misleading and even erroneous results! Simulations help in selecting suitable input sets and design parameters.

Model Building

Engineers are used to building physical models called 'scale models'. Aircraft models (1/25 of actual size) for instance is built to study the space requirements and arrangement of objects in an aircraft. Likewise, ships and buildings are built by naval and civil architects. While these are useful, they are, in most cases, static models. We are often interested in the dynamic performance of systems, before building the actual prototypes – for instance, a model aircraft with sensors attached to the wings and the body for a wind tunnel experiment.

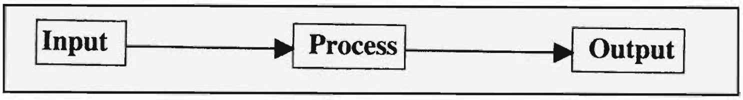

Figure 1. A simple process model.

The physical or (iconic) models are difficult to build and can be expensive. In this article, we concentrate on mathematical modelling and simulation for improving the performance of systems. The simulations can be easily performed using personal computers (PC) and low cost software packages and tools. They can serve as useful learning experience through student projects.

Models are simplified representations of the real world. Most models can be structured as three-part system (*Figure* 1).

As an example, for a flight performance model, input may be the thrust of the aircraft engine and (initial) all-up weight of an aircraft, the process would be the set of equations of motion and fuel consumption rate, while the output would be the velocity, and range of aircraft. We shall illustrate this with our first example. For this, it is essential to focus first on the output of the model i.e., the results expected and work backwards towards the process and the input segments.

To begin with, models are built with a few algebraic equations and a few parameters or variables. They may provide a general description of the system. For instance, a mathematical model for an automobile can be built on the basis of power of the engine, the mass of the car and the frictional co-efficient of the wheels on the road. More complex models would include the shape of the car, the drag force with reference to air (which is important at higher velocities since this force varies with square of the velocity) and environmental conditions, say the temperature of the air, wind velocity and so on. While the simpler model would be useful for the designer of a passenger car or a truck, the advanced model is essential for the designer of a racecar (*Figure* 2).

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One of the major decisions of a model builder is, then, to choose the variables to be included in the model and to exclude 'more

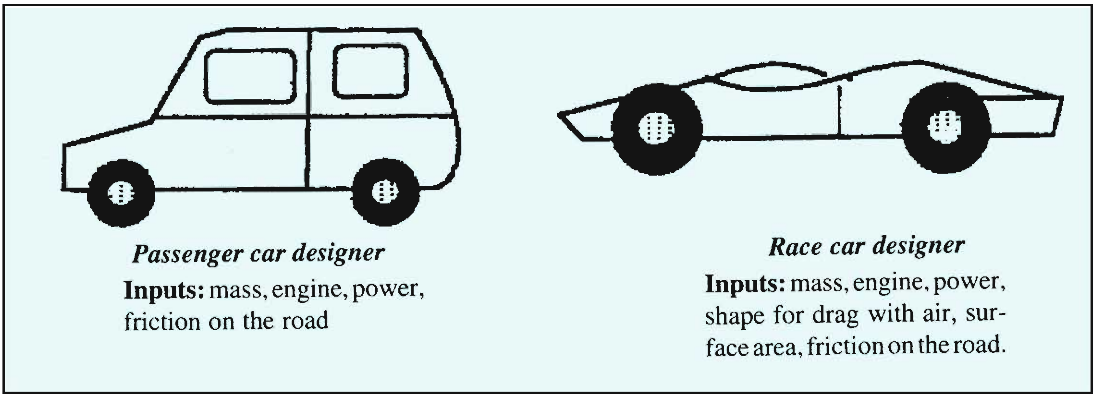

Figure 2. Designer inputs for car models.

complex' or less important variables. A model can grow with addition of variables based on interaction with the users.

Mathematical Structure

Models are built with inter-related set of equations, which may be algebraic, difference or differential equations and with logical statements for constraints and bounds.

To illustrate the model building process, the first example involves the aircraft performance model using only algebraic equations and deterministic variables.

The second example is more complex and involves identifying 'states' of a system and the transition rate from one state to the other. It involves modelling the job market with reference to the school and college-leaving students. It is a probabilistic model.

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In the next part of this article, two more models – 'input/output model' used for production systems or economic studies and a 'discrete event simulation model' are introduced.

Aircraft Performance Model

To illustrate the development of a deterministic, mathematical model, consider the level flight (at constant altitude of an aircraft at subsonic speeds – speed below the speed of sound at constant altitude, i.e., below MACH 1). Our aim is to find an equation for the velocity of the aircraft as a function of time for

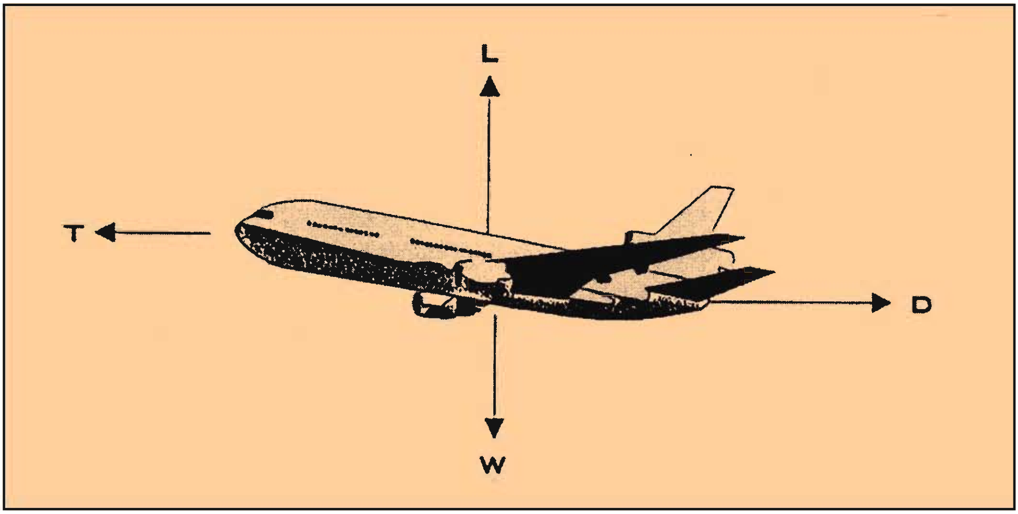

Figure 3. Forces on a cruise flight.

a given thrust level and altitude. We can first analyse the forces acting on the aircraft (*Figure* 3). The lift force L is balanced by the weight of the aircraft W.

$$L = W.$$

The thrust developed by the engine T is equal to drag force D.

$$T = D.$$

[We neglect the small angle between the thrust vector due to engine position and the direction of the drag force. We also neglect the moment caused by the separation of aerodynamic centre, the point at which the lift force acts and the centre of gravity through which the weight acts.]

Now the expressions for lift and drag forces are as follows:

$$L = \left(\frac{1}{2}\rho V^2\right) C_L * S$$

$$D = \left(\frac{1}{2}\rho V^2\right) C_D * S,$$

where ρ is the air density at that altitude, V is the velocity and $Q = 1/2\rho V^2$, the dynamic pressure, C_L and C_D are lift coefficient

Both lift and drag increase with the square of the velocity.

and drag coefficient respectively and S is the lift generating area, essentially the wing area.

Note that both lift and drag increase with the square of the velocity. Since for level flight $D = T$, we replace D by the 'thrust available', i.e., T, (This again is not a constant but depends on the rpm of the engine and the altitude). We equate L with weight since weight is the most important factor for an aircraft. Note that the weight keeps decreasing while the aircraft is in flight due to fuel consumption. Let the rate of decrease in weight be constant.

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -k$$

[k depends on the thrust level of the engine]

We employ an empirical equation called 'drag polar' relating C_L with C_D.

$$C_D = C_{D_0} + KC_L^2$$

Where C_{D_0} and K are constants. We shall use later the following 'drag polar' equation:

$$C_D = 0.025 + 0.035\, C_L^2 .$$

With this collection of equations, we can write the model equations:

$$W = L = \left(\frac{1}{2}\rho V^2\right) C_L S$$

or

$$C_L = \frac{\frac{W}{S}}{\left(\frac{1}{2}\rho V^2\right)} \qquad (1)$$

$$T = \left(\frac{1}{2}\rho V^2\right) S\, C_D$$

$$= \left(\frac{1}{2}\rho V^2\right) S \left(C_{D_0} + KC_L^2\right) \qquad (2)$$

Substituting for C_L in (2) from (1), and solving for V, we can get the velocity V for a given weight. Therefore we can find velocity as a function of time, as weight varies with time. Note that at a constant thrust, the aircraft will accelerate in level flight due to decrease in weight.

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The air density ratio (where ρ/ρ_0 is the air density at altitude (meters) and ρ_0 is air density at sea level) is given by the approximate relation:

$$\sigma = \frac{\rho}{\rho_0} = \exp\left(-\frac{h}{\beta}\right),$$

where $\beta = 9296$ m (for altitudes less than 11 km). For example, for an altitude of 8000 m,

$$\sigma = e^{-\frac{8000}{9296}} = 0.4229\cdot$$

Let us consider a numerical example: to calculate the velocity of a trainer aircraft with initial (all-up) weight of 10,000 kg and wing area 6m^2 at altitude of 8000 m. The engine thrust is taken as 1450 N.

The air density at 8 km $= 0.429 \times 1.225$ kg/m^2
$= 0.5255$ kg/m^2.

Substituting the numbers and solving for velocity V, we get $V =$ 80 m/s or 288 Km/hour.

We can write a computer program to calculate the velocity as a function of time as the fuel is consumed. By integrating the velocity with time we can obtain the range of distance travelled for a given fuel weight. We can also find the optimal height for specified conditions. Such models are routinely used by the airline industry.

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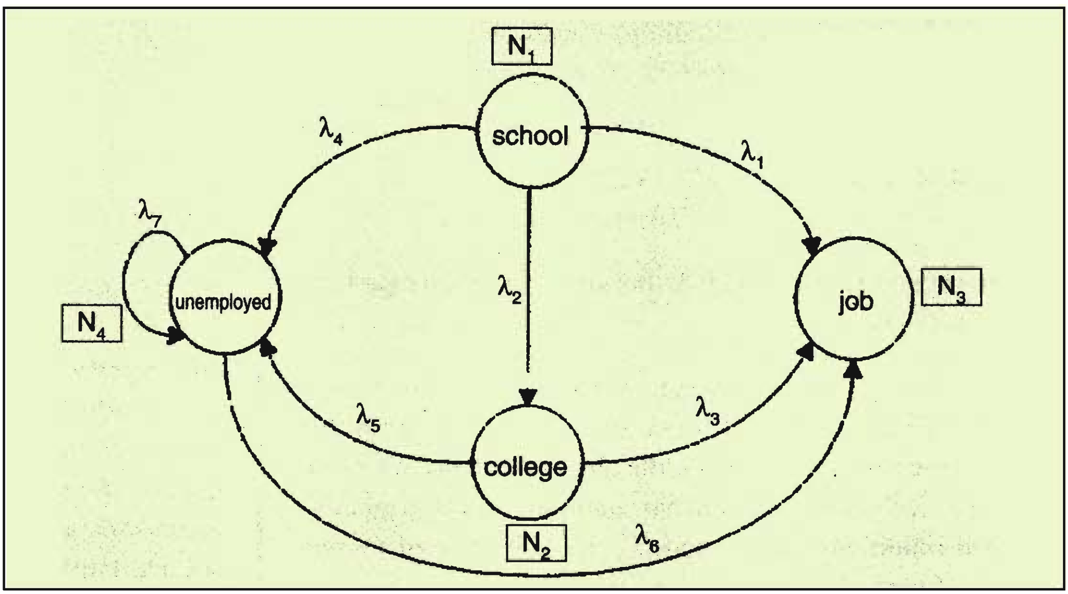
Modelling with Markov Chains

Complex systems can be better understood by developing the system models, which define the states of the system and the process of the system changing from one state to another. Consider a familiar job market system involving schools and colleges. Students who leave the schools (sub-system) may enter colleges for higher studies or enter the job market for employment. The planners in government and industries would like to know how many youngsters would opt for education or jobs. What is important in such models is the rate of change from one state to another, i.e., from high school to workers state or to college students state. The changes are represented by state transition diagrams (STD). Note that college students also seek jobs at a later period. Therefore, the transition from 'college' to 'jobs' states is also indicated. (*Figure* 4).

The rates of transition can be denoted in the STD. As an example, λ_1 = 10% per year, λ_2 = 40% per year and λ_3 = 29% per year.

Figure 4. Markov state transition diagram.

The model developed so far is incomplete, since we have not considered one important state – the unemployed category of



students. The 'unemployed' may be the pool of students who have completed school or college.

The STD for the present model is shown in *Figure* 4. Since the school students have to end up in one of the states in the set: (college, jobs or unemployed).

$$\lambda_1 + \lambda_2 + \lambda_4 = 100$$

Likewise $\lambda_3 + \lambda_5 = 100$ and $\lambda_6 + \lambda_7 = 100$

The dynamic situation, with a time period of one year, arises because the number of school students and the number of college students at the final years would keep varying. Let N_1 and N_2 be the number in each state who are considering for transition i.e., the final year students in school and college, N_3 and N_4 are the numbers joining the ranks of employed and unemployed category, respectively. A large number may remain unemployed and this is shown as a closed loop with λ_7.

The change (per year) is written as first order differential equations, known as Kolmogorov–Chapman equations (see *Box* 1) are given below:

$$\frac{dN_1}{dt} = -(\lambda_1 + \lambda_2 + \lambda_4)N_1$$

$$\frac{dN_2}{dt} = \lambda_2 N_1 - (\lambda_3 + \lambda_5)N_2$$

$$\frac{dN_3}{dt} = \lambda_1 N_1 + \lambda_3 N_2 + \lambda_6 N_4$$

$$\frac{dN_4}{dt} = \lambda_4 N_1 + \lambda_5 N_2 - \lambda_6 N_4$$

These equations can be solved by several methods – for instance using Laplace transforms.

It is useful to find the steady state solution by setting the rates equal to zero i.e.,

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = \frac{dN_3}{dt} = \frac{dN_4}{dt} = 0$$

Box 1. Markov Chains.

The basic theory of transition from one state to another was developed by the Russian mathematician Andrei Markov and hence the name Markov chains.

Andrei Markov [1856-1922] was a student of P L Chebychev and belonged to the illustrious group of St Petersburg School in Russia along with Kolmogorov. Kolmogorov started his career as an engineer and became an applied mathematician. He is credited with applying set theory to probability concepts.

This represents a condition in which the numbers (the population in each state) do not change with time. We might be interested in the number of persons employed/unemployed when the steady state is reached. We, then, have a set of linear (algebraic) equations to solve. [The state 'job' or the employed state is called an absorbing state i.e., a person who is employed continues to remain employed and does not leave the job.]

The most important value of such models lies in simulation or 'what if' analysis. For instance, what would be the impact of increasing the number of colleges or possible means of reducing unemployment?

We have explained so far two methods of model construction – one involving a set of algebraic equations and the other – a set of differential equations which involves the rate of transition from one state to another. These models help us to 'predict' the behaviour of a system (in this case, aircraft velocity and education/job interaction) as a function of time. In Part 2, models using matrix methods and probabilistic approaches will be illustrated. We will also illustrate Monte-Carlo simulation by drawing inputs with random numbers. Such methods can easily handle complex situations, which cannot be solved by analytical means.

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Suggested Reading

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