Crises

Critical Junctures in the Life of a Chaotic Attractor

N Ananthkrishnan and Tuhin Sahai

This article uses the logistic map as an example to illustrate the vagaries that afflict a chaotic attractor in the course of its progress along a path of varying parameter. Several critical junctures, called crises, are encountered, where the chaotic attractor either goes boom or bust!

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Some History

It is believed that Poincaré, a little more than a hundred years ago, was the first to realise the phenomenon of chaos when he studied the 3-body problem in celestial mechanics (see [1]). After a long hiatus, during which other dramatic developments occupied the minds of physicists, chaos reared its head again, this time in fluid mechanics. This occurred first in 1963 in the numerical work by Lorenz on an extremely simplified model of thermal convection. Lorenz observed seemingly irregular (non-steady and non-periodic) behavior even though he had considered a deterministic (non-random) system. Then, in 1971, Ruelle and Takens invoked Lorenz's observation to explain the onset of turbulence in fluids. As yet, nobody had coined the term 'chaos' for this sort of behavior, nor did it capture the imagination of scientists, leave alone the public, at that stage. Many scientists believed that 'chaos' could only be observed in complicated dynamical systems, and that it could not happen in the simple, idealized models generally considered by them.

The age of chaos possibly dawned in 1976 with an article by May who showed that extremely simple dynamical systems could exhibit chaotic behavior. May discussed first-order difference equations and, in particular, he considered the logistic equation,

$$x_{n+1} = f(x_n) = a\,x_n(1 - x_n),$$

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which has since been extremely popular in the chaos literature. May observed that, beyond a critical value of the parameter a, there are "an infinite number of fixed points ... , and an infinite number of different periodic cycles. ... also an uncountable number of initial points which give totally aperiodic trajectories." The words 'chaos' and 'chaotic' appear several times in May's article, and their introduction is attributed by May to a paper by Li and Yorke in 1975. Almost overnight, scientists from various fields of science and engineering feared that they might 'have' chaos right in their backyard. This set off a flurry of activity, notwithstanding some confusion as to the exact definition of chaos.

It was 10 years later that Crutchfield, Farmer, Packard and Shaw wrote a widely accessible article that provided a useful definition of chaos and summarized the developments in the science of chaos till then. At about the same time, James Gleick's delightful book hit the stands, and perhaps did more to popularize chaos and bring it into the drawing rooms than any other. The following years saw the growth of chaos in more ways than one! While it stirred up genuine public interest, for example, in predicting the stock market or forecasting the weather, some scientists made hay by 'discovering' chaos in every other thing under the sun (and within the solar system as well!).

Chaos Today

Chaos today has matured as a science, though evolving still. There is much more agreement on what exactly constitutes chaos. A working definition of chaos with its three main ingredients (in italics), as given by Strogatz, reads as follows: "Chaos is *aperiodic long-term behavior* in a *deterministic* system that exhibits *sensitive dependence on initial conditions*." This type of behavior has been observed experimentally and in mathematical models of dynamical systems from different fields of science and engineering. In most systems, chaos is undesirable, and scientists have devised techniques to control chaotic behavior and force chaotic systems to behave in a regular periodic fashion. However, scientists have also discovered that chaotic systems can be useful, for example, in cryptology, for sending and receiving secret messages. Thus, chaos is not just a fascinating mathematical curiosity, but a development with practical applications in several disciplines.

Chaos also raises some very fundamental questions about the way physics has been understood and done in all these years since Newton and Laplace. Laplace had once boasted that given the position and velocity of every particle in the universe at some instant of time, he could predict the future for the rest of time. Chaos laid to rest Laplace's claim of predictability. Chaotic systems, which follow Newton's laws of motion and have no random influence, are yet unpredictable because small, inevitable errors in the initial conditions can result in solutions that have no correlation with the actual solution from the intended initial conditions. Putting it in other words, the fastest way to predict the state of a chaotic system at a future point of time is to wait for the system to actually get there! This effectively rules out long-term weather prediction, assuming, of course, that variations in the weather can be modeled as a chaotic dynamical system.

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Before the advent of chaos, it was believed that complex behavior required complicated equations. Chaos has shown that low-order deterministic systems with few simple nonlinearities could yield extremely complicated behavior. This is worrying in a way because it means that a whole lot of simple nonlinear systems that occur in various branches of science and engineering could be potentially chaotic. On the other hand, it also gives hope that phenomena like turbulence in fluids that were believed to be extremely complex could perhaps be described in terms of the chaotic behavior of a low-dimensional subsystem. However, there has been little success in this direction so far.

Remarkably, for such a revolutionary theory, many of the developments in chaos are easily accessible to undergraduate students in science and engineering.[1] A study of chaos usually requires an

The first author has been teaching an elective on *Dynamics and Bifurcations* to undergraduate students at IIT, Mumbai in their final year.

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understanding of the geometric theory of nonlinear dynamical systems and attractors in phase space. Bifurcation theory describes the way in which attractors are created and destroyed when a system parameter is varied. The various sequences of bifurcations through which a chaotic attractor can be created are usually called 'routes to chaos.' The necessary condition for chaotic behavior is the presence of a chaotic attractor. The most common method to identify chaos, for example in data collected from an experiment or a numerical simulation, is to compute the Lyapunov exponent. A positive Lyapunov exponent indicates chaos. These topics are well covered in some of the references listed in the *Suggested Reading*. However, no prior knowledge of any of these topics is necessary to understand the rest of this article.

Crisis

One of the central problems in chaos theory has been to decipher the sequence of bifurcations through which a particular system enters into chaotic behavior when a parameter is varied. For example, onset of chaos in the logistic map takes place through a period-doubling bifurcation sequence as described in *Box* 1. There are several articles and textbooks including some listed in the *Suggested Reading* that provide an excellent discussion of the routes to chaos. Of equal importance is the question of what

Box 1. Period-doubling Route to Chaos for the Logistic Map.

Consider the bifurcation diagram in *Figure* 1 for the logistic map for a range of values of the parameter a between 3.5 and 4. Near a=3.5, there are four solutions representing a period-4 cycle, i.e., x alternates between these four values regularly and periodically. As the parameter a is increased, at one point, the period-4 attractor disappears and is replaced by a period-8 attractor. The logistic map shows a sequence of such period-doubling bifurcations. At each period-doubling bifurcation, a period-n attractor vanishes and is replaced by a period-$2n$ attractor. This sequence quickly converges and by $a \approx 3.57$ one can see an infinite period (essentially aperiodic, and hence chaotic) orbit. (The rate of convergence of this sequence and its universal property was first established by Feigenbaum in a result of far-reaching significance.) Thus, the birth of the chaotic attractor in the logistic map takes place through a period-doubling route.

happens to the chaotic attractor further down the (parameter) road? Does the chaotic attractor continue to exist once created, or is it doomed to perish? That is, once a system becomes chaotic, does it continue to show chaotic behavior for further variation of the parameter, or can it revert to non-chaotic behavior? An answer to this question leads us to some very interesting results that could have potential applications to the problems of control and synchronization of chaotic systems.

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In this article, we shall be interested in what happens to a chaotic attractor once it is born after following one of the routes to chaos. We will see that the chaotic attractor faces several critical stages, appropriately called crises, where it undergoes sudden changes as a function of the parameter. At a crisis event, the chaotic attractor either vanishes, or increases in size. The phenomenon of crisis in chaotic attractors was first pointed out by Grebogi, Ott and Yorke. Today, we recognize three different types of crises.

- **Attractor merging crisis**, where a multi-piece chaotic attractor merges together to increase in size smoothly.
- **Interior crisis**, where a chaotic attractor increases in size abruptly.
- **Boundary or exterior crisis**, where a chaotic attractor suddenly vanishes.

The logistic map, besides being simple to understand, shows all three types of crises. We shall therefore use the logistic map to illustrate the phenomenon of crisis in this article. However, crisis phenomena do occur in nearly all chaotic systems. Before proceeding further, let us identify these crises in the bifurcation diagram of the logistic map in *Figure* 1.

Referring to *Figure* 1, the most prominent attractor-merging crisis occurs at $a \approx 3.68$ where the two-piece chaotic attractor merges smoothly at a wedge-shaped point, sometimes called the Miesurewicz point. Several less prominent attractor-merging crises can be noticed between this point and the point of onset of

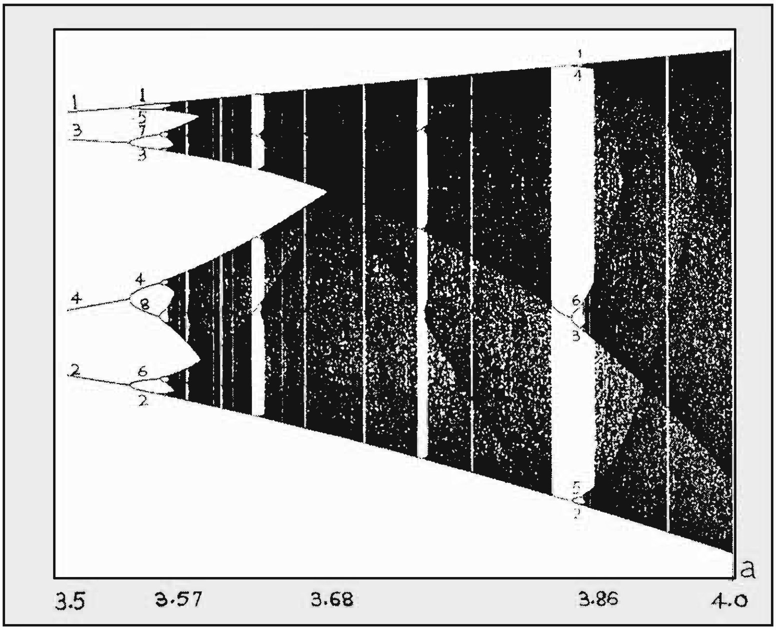


Figure 1. Bifurcation diagram for the logistic map between a=3.5 and 4.0. The numbers inserted in the figure are iterate labels for the periodic orbits.

chaos at $a \approx 3.57$. The most notable interior crisis in *Figure* 1 is near $a=3.86$ in the period-3 window where a 3-piece chaotic attractor abruptly blows up to fill the entire space between the three pieces. Similar interior crises occur at other periodic windows but are not visible in the magnification offered by *Figure* 1. The only example of a boundary crisis in the logistic map occurs at $a=4$ beyond which the chaotic attractor vanishes for good, and all orbits tend to infinity. In the rest of this article, we shall seek to understand what causes these crisis phenomena.

Attractor Merging Crisis

As described in *Box* 1, the onset of chaos in the logistic map occurs through a sequence of period doubling bifurcations at $a \approx 3.57$. Let us look at the newly formed chaotic attractor which can be seen from *Figure* 1 to be made up of many unconnected pieces. With increasing a, the chaotic attractor grows smoothly in size and goes through a sequence of attractor merging crises. At each attractor-merging crisis, the number of pieces of the chaotic attractor is halved, and finally a one-piece chaotic attractor is formed at the crisis point $a \approx 3.68$.

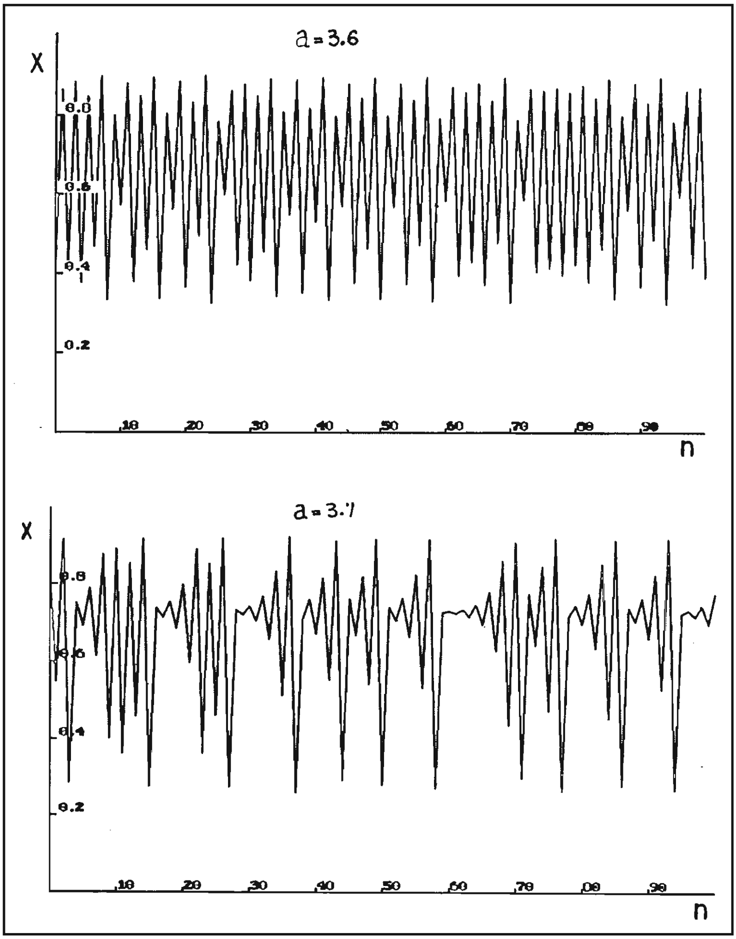


Figure 2. Plot of 100 successive iterates on the chaotic attractor of the logistic map at a=3.6 before the attractor merging crisis and a=3.7 after the attractor merging crisis.

Let us concentrate on this last attractor-merging crisis point. At this point, a 2-piece chaotic attractor combines to form a single connected attractor. It is important to realize that there is order even within the chaos of the 2-piece attractor. The orbit of the chaotic attractor visits the two pieces alternately. If one were to start with a point belonging to the upper piece of the attractor, the next iterate would fall on the lower piece, and the next on the upper one, and so on. This can be verified from the plot of 100 successive iterates on the 2-piece chaotic attractor at $a=3.6$ shown in *Figure* 2. Thus, there is a periodicity of sorts, between the two pieces of the chaotic attractor as a whole, though not between individual points. Considering the two pieces as intervals, each interval maps into the other. We, therefore, have a periodic mapping of the intervals! This is true for all the multi-piece chaotic attractors in *Figure* 1 – the more the pieces, the higher the period. In contrast, the plot of 100 successive iterates on the single-piece chaotic attractor at $a=3.7$ after attractor merging in *Figure* 2 does not show any such pattern.

There is further evidence of order hidden within the chaos in *Figure* 1 which we shall exploit in order to study this crisis event. One can notice several dark curves cutting across the chaotic regions in *Figure* 1. These curves provide the key to understanding crisis once we recognize them as iterates of the point $x=0.5$. The reader can convince herself that this is indeed the case by looking at the shape of the logistic map, which has a single peak at $x=0.5$. If one were to plot the iterates of 0.5 as a function of a for values of a between 3.5 and 4, those curves would closely resemble the curves in the periodic regions in *Figure* 1, and would be identical to the curves in the chaotic regions in *Figure* 1. It is simpler and, for our purpose, adequate to pretend that the dark curves[2] running from left to right in *Figure* 1 are actually the iterates of 0.5, although that is only an approximation in the periodic regions. With this assumption, it is easy to label the dark curves with the iterate numbers of the orbit starting at 0.5, as indicated in *Figure* 1. For example, at $a=3.5$, the iterate labels from top to bottom are 1–3–4–2. And in the period-8 region, the iterates run as 1–5–7–3–4–8–6–2. This is not a difficult exercise, and the reader can work this out without really needing to do any computations.

[2] The dark curves cutting across *Figure* 1 are iterates of the point x=0.5, and are the key to understanding crisis.

Tracing these curves to the attractor merging crisis near $a=3.68$, the third, fourth, fifth, and actually, all the higher-iterate curves can be seen to meet at this point. (It is an easy exercise to reason out that, if the n-th and $n+1$-th iterate curves intersect at a point, for some n, then sc [illegible] t all the higher iterates.) Let us denote by p the value of x at this point. Then, in this case, the orbit from 0.5 appears as follows:

$$0.5 \mapsto f^1(0.5) \mapsto f^2(0.5) \mapsto p \mapsto p \mapsto p \ldots$$

Thus, the point p is a fixed point (period-1 cycle). By computing the first derivative of the logistic map at this point $|f'(p)|$, it is possible to show that this fixed point is unstable. Thus, orbits starting from points near p will diverge from p. Nevertheless, the orbit from 0.5 ends up and stays at p from the third iterate onwards. Such orbits are called *eventually periodic orbits*. Thus, the attractor-merging crisis corresponds to a collision between the 2-piece chaotic attractor and an eventually periodic orbit that ends up at the period-1 point p. This is symbolically depicted in *Figure* 3. (Another way of visualizing this attractor merging crisis is to use cobweb plots as shown in *Box* 2. The symbols appearing in *Figure* A of *Box* 2 are reproduced in *Figure* 3 for ease of correlation between the figures.) Other attractor merging crises are due to similar collisions between a multi-piece chaotic attractor and an eventually periodic orbit of appropriate period.

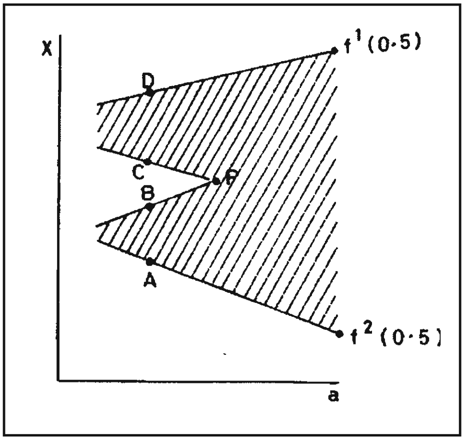
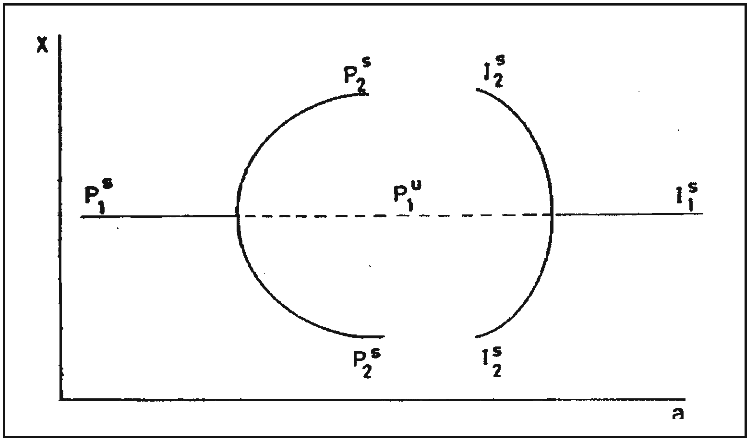


Figure 3. Schematic diagram of the attractor merging crisis. p is the unstable period-1 point with which the 2-piece chaotic attractor collides.

Where do the unstable periodic cycles, for example, the period-1 point p at $a \approx 3.68$, come from? The answer is that they are created at each of the period doubling bifurcations on the route to chaos. This can be better depicted if we introduce some 'symbology'. Let us denote a stable period-n cycle as $\mathbf{P}^s_n$ and an unstable period-n cycle as $\mathbf{P}^u_n$. Then, the period-doubling bifurcation where the stable period-1 cycle $\mathbf{P}^s_1$ loses stability and gives rise to stable period-2 $\mathbf{P}^s_2$ can be depicted as shown in *Figure* 4. Similarly, let us denote an n-piece chaotic attractor by $\mathbf{I}^s_n$, where $\mathbf{I}$ stands for 'interval,' and n is the periodicity between the intervals. Using this notation, the attractor merging crisis at $a \approx 3.68$ is also depicted in *Figure* 4 symbolically. This is not really a 'bifurcation' since the unstable period-1 point con-

Figure 4. Schematic diagram of the creation of the unstable period-1 point P^u_1 (p in Figure 3) at a period-doubling bifurcation, and its collision with the 2-piece chaotic attractor I^s_2 at the attractor merging crisis.



Box 2. Cobweb Plots for Attractor Merging Crisis.

Cobweb plots are plots of x_{n+1} versus x_n in which the function f and a straight line $x_{n+1}= x_n$ are drawn. Intersections of the straight line with the function curve are fixed points. The iterate of a point x_n can be found by drawing a vertical line at x_n until it intersects the function curve. The point of intersection is x_{n+1}. To find the next iterate, this value of x_{n+1} must be transferred back to the X-axis. This is done by drawing a horizontal line at x_{n+1} until it intersects the straight line $x_{n+1}=x_n$. This procedure is repeated all over to get the next iterate. Thus, the cobweb plot is a sequence of vertical and horizontal lines connecting successive iterates.

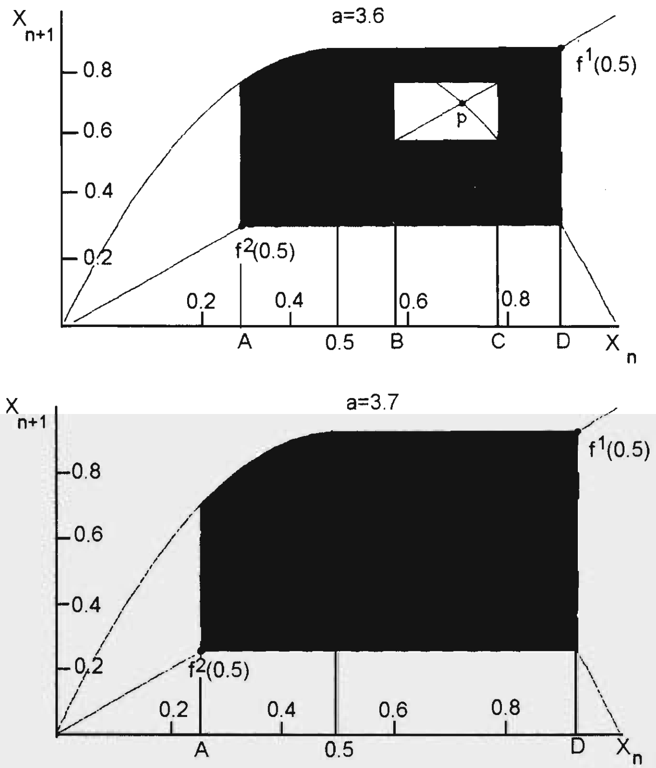


Figure A. Cobweb plots of the chaotic attractor of the logistic map at a=3.6 before the attractor merging crisis and a = 3.7 after the attractor merging crisis.

The attractor merging crisis can be visualized by drawing cobweb plots as in *Figure* A. For a=3.6, the interval AB, which contains the point 0.5, maps into the interval CD. In turn, CD maps into AB. Thus, the first iterate of a point in AB will fall in CD, and the next iterate in AB again. This gives rise to the periodicity between the intervals AB and CD. At the attractor merging crisis, points B and C collide with the fixed point p, and beyond the attractor merging crisis, the periodicity is lost, as can be verified from the cobweb plot for a=3.7 in *Figure* A. These plots were created by using the *Dynamics* software of Nusse and Yorke.

tinues to exist even beyond the crisis. The sketch in *Figure* 4 implies that the unstable period-1 cycle emerges at the period-doubling bifurcation, and continues further to collide with the chaotic attractor at the attractor merging crisis. Similarly, unstable periodic cycles created at each period doubling bifurcation, cause a 'period halving' of the chaotic intervals, later on, at attractor merging crisis points. In a sense, what is created at the period doublings is destroyed at the attractor merging crises, except that one begins with a period-1 attractor and ends up with a single-interval chaotic attractor.

So, what exactly happens at an attractor merging crisis, and why is the orbit starting from 0.5 so significant? We shall use a simple sketch to illustrate what essentially happens at such a crisis. Consider the 2-piece chaotic attractor as sketched in *Figure* 5. We have labelled the boundary points of the two chaotic intervals as U_1, U_2 and L_1, L_2. The point 0.5 lies in the lower interval $L_1 L_2$. The orbit starting from 0.5 maps as indicated by the arrows in *Figure* 5.

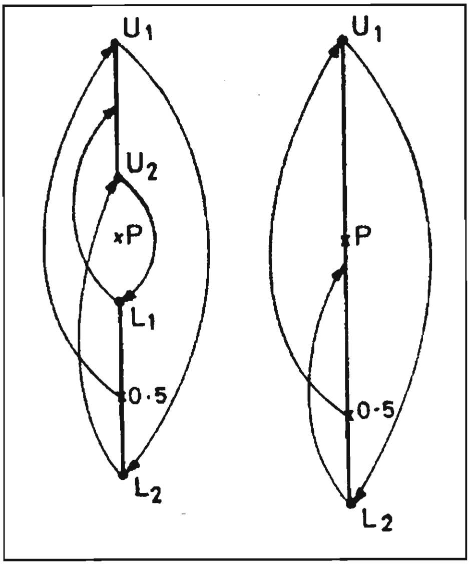


Figure 5. Sketch of the 2-piece chaotic attractor before the attractor merging crisis (left) and the 1-piece chaotic attractor after the crisis (right). Iterates of 0.5 are shown.

$$0.5 \mapsto U_1 \mapsto L_2 \mapsto U_2 \mapsto L_1 \mapsto \ldots$$

L_1 maps inside the interval $U_1\, U_2$. *Figure* 5 implies that the interval $U_1\, U_2$ maps into $L_1\, L_2$, and vice-versa. This is the periodic mapping between the intervals discussed earlier. At the attractor merging crisis, U_2 and L_1 collide with the period-1 point p, and the orbit from 0.5 is eventually periodic at p since successive iterates U_2 and L_1 coincide with p. Before the crisis, L_2 mapped to $U_2 > p$. At the crisis, L_2 maps to $U_2 = p$. Beyond the crisis, L_2 will map to a point below p. Thus, after the crisis, the two intervals not only physically merge into one, but points on one side of p (e.g., L_2) can now map to the same side, as sketched in *Figure* 5. This means that the previous periodicity between the intervals no longer exists, leaving a single chaotic interval.

Interior Crisis

The single-interval chaotic attractor abruptly vanishes at several values of a beyond the attractor merging crisis at $a \approx 3.68$ giving way to various periodic windows. (Each of these windows is created by a saddle-node or tangent bifurcation.) Interestingly, the exact sequence in which these periodic windows are created was predicted by Sharkovskii in the 60's, well before the

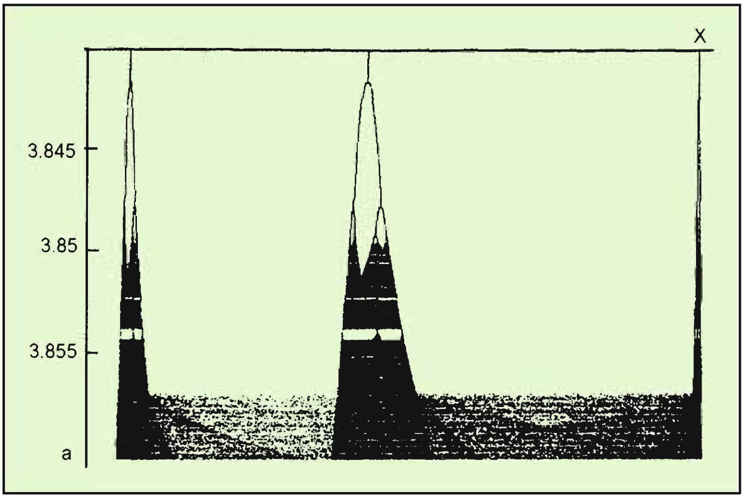


Figure 6. Blow up of the bifurcation diagram for the logistic map in the region of the period-3 window showing the interior crisis at $a \approx 3.86$.

notion of chaos came into vogue. The most prominent of these is the period-3 window at $a \approx 3.83$. A blow-up of the period-3 window is shown in *Figure* 6. The sequence of bifurcations and crises in *Figure* 6 looks like three miniature copies of the bifurcation diagram in *Figure* 1. Such copies, at all scales, are found within each periodic window, and in windows within windows, and so on. This underlines the fractal nature of chaos [8].

Within a periodic window, each branch of the periodic attractor undergoes a period doubling sequence leading to chaos. The pieces of the chaotic attractor in each branch then merge via attractor merging crises to form a single-piece attractor in a replay of the picture discussed in the previous section. Thus, each branch in the period-3 window contributes one piece to the overall 3-piece chaotic attractor. Interestingly, these pieces when considered as three intervals show period-3 behavior, i.e., every third iterate falls back in the same interval. However, unlike the attractor merging crisis, these three pieces do not merge smoothly. Instead, they suddenly join together to form a single-piece chaotic attractor at an interior crisis at $a \approx 3.86$.

Once again, we track the iterate curves of 0.5 across *Figure* 1 till we reach the interior crisis. At the crisis point, the seventh iterate curve of 0.5 coincides with the fourth, the eighth with the fifth, the ninth with the sixth, and so on. Thus, the orbit from 0.5

is an eventually periodic orbit that ends up at an unstable period-3 cycle. If these periodic points are labelled p_1, p_2, p_3, then the orbit from 0.5 appears as follows:

$$0.5 \mapsto f^1(0.5) \mapsto f^2(0.5) \mapsto f^3(0.5) \mapsto p_1 \mapsto p_2 \mapsto p_3 \mapsto p_1 \cdots$$

The interior crisis, therefore, occurs at a collision between the 3-piece chaotic attractor and an eventually periodic orbit of period 3, as sketched in *Figure* 7. The genesis of the unstable period-3 cycle at the crisis can be traced to the tangent bifurcation that created the period-3 window. At the tangent bifurcation, the single-piece chaotic attractor gives way to a stable period-3 cycle and an unstable period-3 cycle created simultaneously. The unstable period-3 cycle eventually collides with the 3-piece chaotic attractor to give back the single-interval chaotic attractor at the crisis. Before the interior crisis, the three intervals of the chaotic attractor would map into the others cyclically. This periodicity is lost beyond the interior crisis, once the boundary points of the three intervals collide with the periodic points p_1, p_2, p_3, in a fashion similar to that seen in the case of an attractor merging crisis.

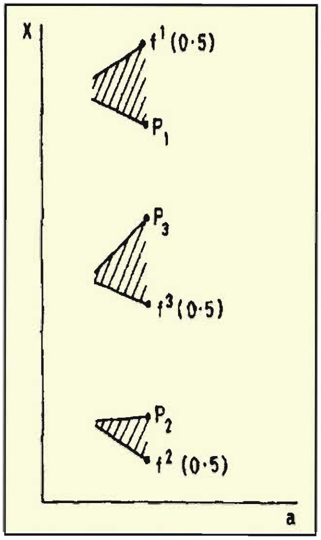
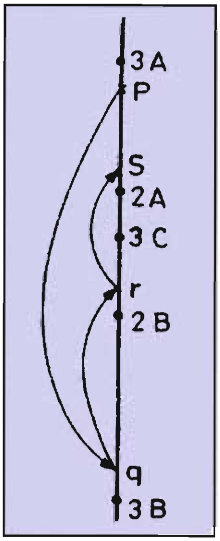


Figure 7. Schematic diagram of the interior crisis in Figure 6. p_1, p_2, p_3 are the unstable period-3 cycle with which the 3-piece chaotic attractor collides at the interior crisis.

By the time the logistic map reaches the period-3 window, it has collected unstable periodic cycles of all periods, including multiple periodic cycles of the same period, at various period doubling and tangent bifurcations. In fact, the period-3 cycle is the last to be added to this collection. This is, of course, a part of the remarkable result derived by Sharkovskii. One can think of a chaotic orbit as a ball being bounced around by a collection of unstable periodic cycles. A few orbits that start exactly on one of the unstable periodic points will show periodic behavior. For the vast majority of orbits that start off the unstable periodic points, future iterates are dictated by the nearest unstable periodic point. As an illustration, consider just two unstable cycles of periods 2 and 3, respectively, with points labelled *2A*, *2B* and *3A*, *3B*, *3C* in *Figure* 8. A chaotic orbit that starts at p near 3A will get mapped to q near *3B* and then to r near *3C*. But, point r lies nearer *2B* than *3C*, and the next iterate s will fall nearer *2A* than

Figure 8. Sketch of a chaotic orbit starting at 'p' being 'bounced' from one periodic cycle to another.



When the collection of unstable periodic cycles has cycles of every period, a chaotic orbit could potentially show every possible near-periodic behavior.

3*A*. That is, the chaotic orbit has been captured from the period-3 by the period-2 cycle. A chaotic orbit can, in effect, be looked upon as a sequence of near-periodic behaviors dictated by such 'captures.' When the collection of unstable periodic cycles has cycles of every period, a chaotic orbit could potentially show every possible near-periodic behavior. This has led to the coining of slogans like 'period three implies chaos,' although, as we have seen, chaos can very well exist before the unstable period-3 cycle is created.

Boundary Crisis

The logistic map has two fixed points (period-1 cycles), one at $x=0$ and the other at $x=1-1/a$. At the transcritical bifurcation between these two fixed points at $a=1$, the point at $x=0$ loses stability while the other point at $x=1-1/a$ becomes stable. All the unstable periodic cycles in our discussion so far have grown out of bifurcations of the second fixed point at $x=1-1/a$. Collisions between the chaotic attractor and these unstable cycles led to attractor merging and interior crises. In both these types of crises, the colliding periodic points were within (though not belonging to) the basin of attraction of the chaotic attractor. That is, the periodic points would always lie in the *interior* of the chaotic intervals, e.g., within $U_1 L_2$ in *Figure* 5.

The first fixed point at $x=0$, unstable for $a>1$, did not play any part in the proceedings so far. However, as the single-interval chaotic attractor grows in size after the interior crisis at $a \approx 3.86$, it meets with the $x=0$ period-1 cycle at $a=4$. This is a boundary crisis beyond which the chaotic attractor abruptly disappears. Following predictable lines, we trace the iterate curves of 0.5 in *Figure* 1 all the way to the right of the figure, and notice that the second and higher iterates all converge at $x=0$. Thus, the orbit from 0.5 for $a=4$ is an eventually periodic orbit of period 1 ending up at the unstable fixed point at zero. This orbit looks like

$$0.5 \mapsto f^1(0.5)=1 \mapsto 0 \mapsto 0 \mapsto \ldots$$

The boundary crisis occurs at a collision of the chaotic attractor with this eventually periodic orbit. This is sketched in *Figure* 9. The period-1 point $x=0$ lies on the *boundary* of the single-piece chaotic interval. After the crisis, points belonging to the erstwhile chaotic attractor can map outside the interval (0,1) via points in the vicinity of 0.5 that map to points greater than 1. All such points find their way to infinity. Except for a few orbits that start precisely on the unstable periodic points, all other orbits end up at infinity and the chaotic attractor no longer exists.

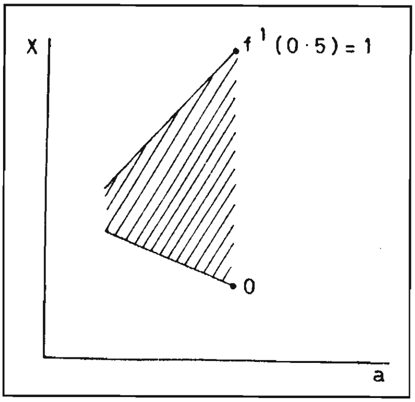


Figure 9. Schematic diagram of the boundary crisis at a=4. The 1-piece chaotic attractor collides with the unstable period-1 point at '0.'

Acknowledgments

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Suggested Reading

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