Exchange of identical particles is one of the theoretical ideas in quantum physics which was consistent with scattering experiments. There are subtle differences in the exchange phenomenon in a two dimensional plane as compared with our three dimensional world. This leads to the presence of a new class of particles called *anyons* besides the two well-known universal categories (bosons and fermions) in the two dimensional plane.

1. Introduction

Exchange of identical particles is a simple mathematical technique which accounts for the presence of *two classes* of particles in our three-dimensional world – namely, *bosons and fermions*. In fact, the above theory confirms the experimental observation that the system of identical helium atoms He$_3$ behaves differently from the system of identical He$_4$ atoms (isotope) in a scattering process.

Suppose we restrict the exchange to be performed only in a two dimensional plane, we will see that *one more class of particles* is allowed to exist. Wilczek coined the name *anyons* for such exotic particles (not seen in our real world). The reader may wonder as to why it is interesting to look at *two dimensions* which is far removed from our existing world. What actually happens is that some physical systems, though three-dimensional, behave as two-dimensional systems when subjected to external forces. For instance, electrons in a cubic semiconductor get confined to a thin two-dimensional layer by
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the application of strong electric fields.

Initially, anyons were considered to be only mathematical fantasies. But this view drastically changed when these theoretically predicted anyons were experimentally identified as collective excitations in certain condensed matter systems. This observation certainly had a great impact resulting in widespread interest in anyon physics. For example, the collective excitations above ground state in systems exhibiting fractional quantum Hall effect (for details on Hall effect, see [3]) were identified with anyons. In fact, the theoretical work of Laughlin confirming the experimental predictions of a new form of quantum fluid with collective excitations by Störmer and Tsui bagged them the Nobel Prize in Physics in 1998. Another area in which anyons are conjectured to play a role is in the theory of high temperature superconductivity (for details see [4]) but no concrete experimental evidence has been observed till date.

With so many interesting applications of anyon physics, it is very essential to learn how these anyons emerge as a logical possibility only in a two dimensional space. This is what we elaborate in this article using exchange phenomenon. We will not describe applications of anyons as it is beyond the scope of this article.

The plan of this article is as follows: In section 2, we will recapitulate the textbook material on exchange of identical particles in three dimensions. In section 3, we shall present exchange phenomenon in the two dimensional plane and show how a new class of particles called anyons emerges as a logical possibility. We summarise the results in the concluding section.

2. Exchange of Identical Particles in Three Dimensions

It is well known that there are only two classes of particles in our three-dimensional world, namely, bosons and
fermions. We shall use the simple technique of exchange phenomenon to account for the two classes.

Let us consider a system of \( n \) identical particles. In quantum physics, the state of \( n \) identical particles is described by a wave function \( \Psi(r_1, r_2, \ldots, r_n) \) where \( r_1, r_2, \ldots, r_n \) symbolically denote position and other quantum numbers of particles 1, 2, \( \ldots \), \( n \), respectively. In simple terms, a wave function is a complex function whose modulus squared \( |\Psi(r_1, r_2, \ldots, r_n)|^2 \) gives the probability density of finding these \( n \) particles with the given position and other quantum numbers.

What happens when we exchange particle 1 with particle 2? The new configuration will be described by the wave function \( \Psi(r_2, r_1, \ldots, r_n) \). Since the particles are identical, the physical state of the system given by modulus squared \( |\Psi(r_1, r_2, \ldots, r_n)|^2 \) should be unchanged. Hence, the quantum state \( \Psi \) can at most pick up a phase under exchange. That is,

\[
\Psi(r_2, r_1, \ldots, r_n) = e^{i\theta} \Psi(r_1, r_2, \ldots, r_n) \tag{1}
\]

Repeating this exchange, we get back the original quantum state:

\[
\Psi(r_1, r_2, \ldots, r_n) = e^{i\theta} \Psi(r_2, r_1, \ldots, r_n) = e^{2i\theta} \Psi(r_1, r_2, \ldots, r_n) \tag{2}
\]

which implies that \( e^{2i\theta} = 1 \) giving two possibilities; \( \theta = 0 \) or \( \theta = \pi \). These two possibilities represent the two classes of particles. Hence the quantum state for these two classes under exchange of particle 1 and particle 2 satisfy

\[
\Psi(r_2, r_1, \ldots, r_n) = \Psi(r_1, r_2, \ldots, r_n). \tag{3}
\]

\[
\Psi(r_2, r_1, \ldots, r_n) = -\Psi(r_1, r_2, \ldots, r_n). \tag{4}
\]

Suppose particle 1 and particle 2 have the same set of quantum numbers \( r_1 = r_2 \). Clearly \( \Psi(r_1, r_1, r_3, \ldots, r_n) \neq 0 \) for the wave function obeying (3) whereas \( \Psi(r_1, r_1, r_2) \neq 0 \).
\[ r_3, \ldots r_n = 0 \] for the wave function obeying (4). Recalling Pauli's exclusion principle, which states that no two identical fermions can have the same set of quantum numbers, we deduce that (4) denotes the property of fermions under exchange and (3) represents the property of bosons under exchange.

Thus the two universal categories of *bosons and fermions* emerge naturally from the simple exchange process in three dimensions. We will see in the next section that the restriction to two dimensions results in peculiar features.

3. Exchange of Particles in Two Dimensions

In the previous section, we did not specify whether the exchange is clockwise or anticlockwise. However in a two dimensional plane, we will see that *clockwise exchange is different from anticlockwise exchange*. Such a distinction disappears once we have an extra third dimension.

Consider \( n \) identical particles which can only move in two dimensions, say the \( x-y \) plane as shown in *Figure 1*.

Let us denote the clockwise exchange of particle 1 with particle 2 by \( P_{12}^{(+)} \) (as shown by dashed line in *Figure 1*).
As mentioned in the previous section, the physical state remains unchanged under such an operation. Therefore the wave function can at most change by a phase:

$$\Psi(r_1, r_2, \ldots, r_n) \to \mathcal{P}_{12}^{(+)\Psi}(r_1, r_2, \ldots, r_n) = e^{i\nu_{12}}\Psi(r_1, r_2, \ldots, r_n)$$  \hspace{1cm} (5)

where $\nu_{12}$ encodes the nature of the particles. Instead of clockwise exchange, we perform anticlockwise exchange $\mathcal{P}_{12}^{(-)}$ (as shown by dotted line in Figure 1). Let $\beta_{12}$ be the phase picked up by the wave function:

$$\Psi(r_1, r_2, \ldots, r_n) \to \mathcal{P}_{12}^{(-)\Psi}(r_1, r_2, \ldots, r_n) = e^{i\beta_{12}}\Psi(r_1, r_2, \ldots, r_n)$$  \hspace{1cm} (6)

It is an obvious fact that a clockwise exchange of two particles followed by anticlockwise exchange of the same particles is no exchange at all— that is,

$$\mathcal{P}_{12}^{(-)}\mathcal{P}_{12}^{(+)\Psi}(r_1, r_2, \ldots, r_n) = e^{i\beta_{12}}e^{i\nu_{12}}\Psi(r_1, r_2, \ldots, r_n) \equiv \Psi(r_1, r_2, \ldots, r_n)$$  \hspace{1cm} (7)

which implies

$$\beta_{12} = -\nu_{12}$$  \hspace{1cm} (8)

Thus, we see that the phase under anticlockwise exchange is inverse of the phase for the clockwise exchange.

As long as we pretend to live in the two dimensional space (here it is the $x-y$ plane), the clockwise exchange is distinct from anticlockwise exchange and there is no restriction on $\nu_{12}$. It can take any value between 0, and $\pi$. The two extreme values $\nu_{12} = 0$ and $\nu_{12} = \pi$ correspond to bosons and fermions, respectively. The particles with intermediate values $\nu_{12} \in (0, \pi)$ represent anyons.

Now we allow the possibility of movement in the third dimension also. Then the particle will have the freedom
of moving in the z-direction as well. Clearly, the clockwise exchange can be easily made into an anticlockwise exchange by rotating the dotted curve in Figure 1 about the x-axis in the three-dimensional space. This observation automatically restricts the clockwise phase to be equal to anticlockwise phase

\[ e^{i\nu_{12}} = e^{-i\nu_{12}} \]  

The above equation also implies that the exchange of particle 1 and particle 2 twice is no exchange at all in three dimensions giving rise to two possible values \( \nu_{12} = 0 \) or \( \pi \) as obtained in the previous section. Thus, we see that the presence of the third dimension plays a significant role in removing the distinction between clockwise and anticlockwise exchanges of identical particles.

In this section, we have demonstrated that the two-dimensional plane is very special with two types of exchanges – clockwise and anticlockwise giving rise to a new set of particles called anyons besides the well-known universal classes of bosons and fermions.

4. Conclusions

In this article, using the technique of exchange, we have shown that only two categories of particles (bosons and fermions) are allowed in our three dimensional world whereas a new set of particles called anyons are also allowed in a two dimensional plane. This crucial difference stems from the fact that the clockwise exchanges and anticlockwise exchanges are distinct in two dimensions but not in three dimensions.

Acknowledgments

I would like to thank Ameeya, Mohan, Surya Nayak, Umasankar, T R Govindarajan for their comments and suggestions.