Think It Over

This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

Shailesh A Shirali
Rishi Valley School
Rishi Valley 517 352
Andhra Pradesh, India

The Iterated Sine

Here is the solution to the problem posed in the Think-it-over column of Vol.5, No.5, p.90, 2000.

Problem

Consider the function $f_n(x)$ defined for positive integers n and real numbers x by $f_n(x) = \sin(\sin(\sin(. \quad (\sin x))))$, with n applications of the sine function. Thus $f_2(x) = \sin(\sin x)$, $f_3(x) = \sin(\sin(\sin x))$, and so on.

Let $x \in (0, \pi/2)$ be chosen, and let the sequence

$$f_1(x), f_2(x), f_3(x), \qquad f_{100}(x),$$

be computed. It is found that for each fixed x and for all sufficiency large n, we have $f_n(x) \approx \sqrt{3/n}$. How may this phenomenon be explained?

Solution by the proposer Shailesh Shirali

For fixed $x \in (0, \pi/2)$, let $a_n = f_n(x)$. Then we have, inductively,

$$0 < a_n = \sin a_{n-1} < a_{n-1},$$

so the sequence of a_n's is monotone decreasing. Since the sequence is bounded below by 0, it possesses a limit, say L. We have, then, $L = \sin L$ i.e., $L = 0$.

There are now two approaches one may take. The first approach is a bit hair-raising, and we do not recommend it for the faint of heart. However it provides a good heuristic insight into what is happening, and one may hazard a guess that a Poincaré or an Euler would have loved this 'solution'! We write

$$\frac{a_{n+1} - a_n}{1} = \sin a_n - a_n.$$

Thinking of a_n as a function of the *continuous* argument n, and of the '1' as indefinitely small (after all, when n is indefinitely large, surely '1' may be regarded, in comparison, as indefinitely small?), we rewrite the relation on the right as $\Delta a / \Delta n = \sin a - a$, and, in the limit, as

$$\frac{da}{dn} = \sin a - a.$$

We now have a differential equation! Alas, it is not one that possesses a closed-form solution. However we know that $a_n \to 0$ as $n \to \infty$, so for sufficiently large n we may write

$$\begin{aligned} \frac{da}{dn} &= a - \frac{a^3}{6} - a = -\frac{a^3}{6}, \quad \frac{-6da}{a^3} = dn, \\ \frac{3}{a^2} &= n + \text{some constant (by integration).} \end{aligned}$$

Since $a_n \to 0$ as $n \to \infty$, the constant of integration must be 0. We get, therefore, for large n,

$$\frac{3}{a_n^2} \approx n, \quad a_n \approx \sqrt{\frac{3}{n}},$$

as required. (The reader may perhaps now see why we do not recommend this solution for the faint of heart!)

Actually, this approach has excellent heuristic value, and one can use it to conclude, for instance, that:

• the sequence $\{a_n\}$ defined by $0 < a_1 < 1, a_{n+1} = 1n(1 + a_n)$ converges to 0, with $a_n \approx 2/n$ for large n (the reader is invited to supply the details);

• the sequence $\{a_n\}$ with $0 < a_0 < 1$ and $a_{n+1} = a_n - a_n^2$ converges to 0, with $a_n \approx 1/n$ for large n;

• more generally, the sequence $\{a_n\}$ defined by

$$0 < a_0 < 1, \qquad a_{n+1} = a_n - ba_n^k,$$

where $k > 1$ and $0 < b < 1/a_0^k$, converges to 0; and we have, moreover,

$$a_n \approx \left(\frac{1}{(k-1)bn}\right)^{1/(k-1)} \quad \text{for large } n.$$

On the other hand, the sequence $\{a_n\}$ defined by $0 < a_1 < 1, a_{n+1} = \tan a_n$ exhibits 'wild behaviour', and fails to converge. This too may be predicted by our cavalier approach.

Now for the second approach, which is very rigorous and mathematically most satisfying. Consider the quantity b_n defined by

$$b_n = \frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}.$$

Since $0 < a_{n+1} < a_n$ we have $b_n > 0$ for all n. Next, we show that $\lim_{n\to\infty} b_n$ exists, and equals 1/3. We have,

$$\lim_{n\to\infty}\left(\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{\sin^2 a_n} - \frac{1}{a_n^2}\right)$$

$$= \lim_{t\to 0}\left(\frac{1}{\sin^2 t} - \frac{1}{t^2}\right) = \frac{1}{3} \quad \text{(by L' Hospital's rule!).}$$

So $b_n \to 1/3$ as $n \to \infty$. Therefore we can say:

$$\frac{1}{3} = \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{b_1 + b_2 + \quad + b_n}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{a_{k+1}^2} - \frac{1}{a_k^2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{a_{n+1}^2} - \frac{1}{a_1^2}\right) \text{ (by `telescopic' cancellation)},$$

$$= \lim_{n\to\infty} \frac{1}{n a_{n+1}^2} = \lim_{n\to\infty} \frac{1}{n a_n^2}.$$

It follows that for large n we have $na_n^2 \approx 3$, which is equivalent to $a_n \approx \sqrt{3/n}$. □

A somewhat different solution to the problem was received from Ritesh K Singh (student, Centre for Theoretical Studies, I I Sc, Bangalore 560012, India.).

See also the article by V Balakrishnan in this issue (pp.18-27).

"Now I understand how come our ancestors descended from apes. They all sat around in trees."

From: *Gene Antics*