In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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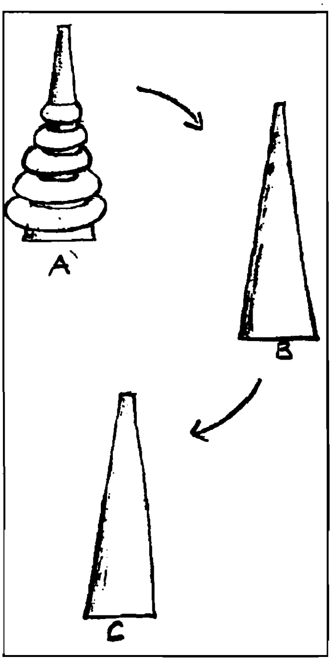
The 'Oriented' Tower of Hanoi



The 'Tower of Hanoi' or 'Tower of Brahma' is well known. We are given n disks (numbered $1, 2, 3, \quad , n$); in size we have $\#1 < \#2 < \#3 < \quad < \#n$. Also given are three pegs A, B and C. The disks are initially all on A. They are required to be moved to C, using B as a halting station. The rules are simple: *a larger disk is not permitted to sit on top of a smaller one.* With these restrictions we are required to move the disks from A to C using the least possible number of moves. It is well known, and easy to prove via induction, that the least number of moves required to do the needful is $2^n - 1$. Note that in this version, if the terminus is B rather than C, nothing essential changes; the formula is the same.

We now impose an additional condition: *the direction of movement must be clockwise.* That is, we may move disks directly from A to B, from B to C, or from C to A; but *not* directly from B to A, from C to B, or from A to C. How does this condition affect the formula for the least number of moves? Clearly now the choice of terminus is no longer irrelevant.

Let u_n be the least number of moves required if the disks are to be moved to B, and let v_n be the corresponding number if the disks have to be moved to C. Note that u and v are not the same. For instance, $u_1 = 1$ whereas $v_1 = 2$; and it may be verified that $u_2 = 5$ whereas $v_2 = 7$. Elementary counting arguments yield the following equations:

$$u_n = 2v_{n-1} + 1, \tag{1}$$
$$v_n = 2v_{n-1} + u_{n-1} + 2. \tag{2}$$

To see why (1) is true, note that before disk $\#n$ can be moved from A, disks $\#1, \#2, \quad, \#(n-1)$ must first be moved; clearly they must all be moved to C, for only then can disk $\#n$ be moved to B. This requires v_{n-1} moves. Then disk $\#n$ is moved from A to B (1 move required), and finally disks $\#1, \#2, \quad, \#(n-1)$ are moved from C to B via A (from the 'back'), taking another v_{n-1} moves and bringing the total number of moves to $2v_{n-1} + 1$; therefore $u_n = 2v_{n-1} + 1$. Equation (2) may be justified along similar lines.

From (1) and (2) we obtain:

$$u_n = 2(u_{n-1} + u_{n-2}) + 3,$$
$$v_n = 2(v_{n-1} + v_{n-2}) + 3.$$

Observe that both sequences obey the same recursion! The initial conditions are:

$$u_1 = 1, \quad u_2 = 5, \quad v_1 = 2, \quad v_2 = 7.$$

A Mathematica program to compute v_n and u_n for various values of n is given below.

```
ClearAll[u, v]; SetAttributes[{u, v}, Listable];
u[1] = 1; u[2] = 5;
v[1] = 2; v[2] = 7;
u[n_] := 2 v[n - 1] + 1
v[n_] := 2 v[n - 1] + u[n - 1] + 2;
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Here are some sample output values:

u[Range[1, 10]] ={1,5,15,43,119,327,895,2447,6687,18271}
v[Range[1, 10]] ={2,7,21,59,163,447,1223,3343,9135,24959}

Observe that for both sequences, each term is roughly three times the preceding term (a bit less, in fact). We shall presently be able to explain why this is so.

Now write z_n for $u_n + 1$ or $v_n + 1$; then we find that z obeys a 'homogeneous' recursion relation:

$$z_n = 2(z_{n-1} + z_{n-2}).$$

The initial conditions are: (a) for the u-sequence, $z_1 = 2$ and $z_2 = 6$; (b) for the v-sequence, $z_1 = 3$ and $z_2 = 8$.

Assuming that z is an exponential function, i.e., $z_n = t^n$ for some suitable number t, we find by substitution into the above relation that $t^2 = 2t + 2$. This equation has the roots $1 + \sqrt{3}$ and $1 - \sqrt{3}$. It follows that the sequence of powers of either of these two numbers obeys the same recursive law as z, and so therefore does any linear combination of the two power sequences. So we write

$$z_n = a(1 + \sqrt{3})^n + b(1 - \sqrt{3})^n$$

for unknown constants a and b, and then find a and b via the initial conditions $z_1 = 2, z_2 = 6$ (or $z_1 = 3, z_2 = 8$ for the v-sequence). This is easy to do, for we have only to solve a pair of simultaneous equations.

Whereas in the unoriented tower of Hanoi problem a very simple formula holds for the least number of moves required, in the present problem the formula is substantially more complicated. We find, after going through the steps suggested above, that

$$u_n = \left(\frac{3+\sqrt{3}}{6}\right)(1+\sqrt{3})^n + \left(\frac{3-\sqrt{3}}{6}\right)(1-\sqrt{3})^n - 1,$$

$$v_n = \left(\frac{3+2\sqrt{3}}{6}\right)(1+\sqrt{3})^n + \left(\frac{3-2\sqrt{3}}{6}\right)(1-\sqrt{3})^n - 1.$$

That may be a bit more than we had bargained for! As a matter of fact, one has also the curious formulae

$$u_n = \sum_{i\geq 0} 2^{n-i}\binom{n-i}{i} - 1$$

$$v_n = \sum_{i\geq 0} 2^{n-i}\binom{n+1-i}{i} - 1.$$

Now we can explain why in both sequences each term is a bit less than three times the preceding term. Observe that $1-\sqrt{3} \approx -0.73205$, which in absolute value is strictly less than 1. So as n gets larger and larger, the contribution of the term $b(1-\sqrt{3})^n$ gets smaller and smaller. Since the quantity alternates in sign, and since u_n and v_n are integers, we arrive at the following:

$$\begin{aligned} u_n &= \text{the integer closest to } \left(\frac{3+\sqrt{3}}{6}\right)(1-\sqrt{3})^n - 1, \\ v_n &= \text{the integer closest to } \left(\frac{3+2\sqrt{3}}{6}\right)(1+\sqrt{3})^n - 1. \end{aligned}$$

Sample verification.

Let $n = 4$; we have

$$\left(\frac{3+\sqrt{3}}{6}\right)(1+\sqrt{3})^4 - 1 \approx 42.9393.$$

$$\left(\frac{3+2\sqrt{3}}{6}\right)(1+\sqrt{3})^4 - 1 \approx 59.0222.$$

Indeed we find that $u_4 = 43$ and $v_4 = 59$. It follows from the above that for large enough n,

$$\frac{u_n}{u_{n-1}} \approx 1+\sqrt{3},\ \frac{v_n}{v_{n-1}} \approx 1+\sqrt{3}.$$

Note that $1+\sqrt{3} \approx 2.73205$. This accounts for the observation made earlier. Now we are able to make a more precise observation: *For sufficiently large n we have:*

$$\frac{u_n}{u_{n-1}} \approx 2.73205,\ \frac{v_n}{v_{n-1}} \approx 2.73205.$$