The Football

1. From Euclid to Soccer it is ...

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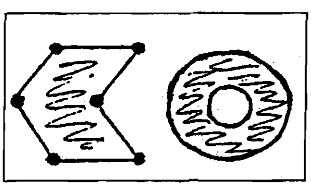
A **football** is a 3-dimensional convex polyhedron with each face a regular pentagon or a regular hexagon and with at least one hexagonal face.

This article is in two parts. In this first part, we will prove that a football exists and is unique and in the second, we identify its group of symmetries. (We will incidentally do similar things for the platonic solids to some of which the football is closely related.) I heard of this problem from Amit Roy of TIFR, Mumbai. The ideas used in the proof of the existence and uniqueness are also his. Most of the other proofs presented here can be found in Gallian (1999) and Coxeter (1948).

A *convex set* is a set $C \subseteq \mathbb{R}^3$ such that $A, B \in C \Rightarrow AB \subseteq C$. (Here AB denotes the segment joining A and B and we will study only convex subsets of $\mathbb{R}^3$.) Intersection of any family of convex sets is convex. There are plenty of examples: the empty set, a point, a line, a line segment, a plane, a half plane, a quadrant, a disc, an elliptic region, a half space (i.e., points lying on one side of a plane), a ball, a pyramid and a prism with a convex base (right or not). See V S Sunder's articles [3] for a discussion on various aspects of convexity. The five platonic solids are convex. The two figures in *Figure* 1 are not. Note that the hexagon shown in the figure is equilateral but not equiangular. By a *regular polygon* we mean a plane polygon which is both equilateral and equiangular.

Figure 1.



A *convex polyhedron* is a finite intersection of closed half-spaces. The disc and the cylinder are not convex polyhedra.

A *convex polytope* is a convex polyhedron which is bounded.

By a *regular solid* we mean a convex polyhedron such that the faces are all regular, equal polygons and the same number of faces occurs at each vertex. It was already known 2400 years ago that there are exactly five such solids, viz. the platonic solids, see *Box* 1. The Greeks associated the tetrahedron (this means a solid bounded by four faces) with fire, the cube with earth, the octahedron with air, the icosahedron with water and the dodecahedron with universe or cosmos. The study of dodecahedron was considered dangerous and restricted during some period. On the other hand, the dodecahedron was used as a toy at least 2500 years ago.

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Apparently Theaetetus "first wrote on the 'five solids' as they are called" around 380 B.C. and probably knew that there are exactly five regular solids. Around 320 BC, Aristaeus (known as 'the elder') wrote a book called *Comparison of the five regular solids.* Euclid wrote his *Elements* around 300 BC.

In the diagrams in *Box* 1, the symbol $\{p, q\}$, known as a Schlafli symbol, means that each face is a regular p-gon and that there are q faces at each vertex. Of the five regular solids, the cube and octahedron are duals of each other, the dodecahedron and the icosahedron are duals of each other and the tetrahedron is self-dual in the following sense: if we start with the cube and form a new solid by taking a new vertex at the centre of each face of the cube and joining two new vertices by an edge iff they are centres of adjacent faces of the cube, we get the octahedron. If we do the same starting from the octahedron we get back the cube; similarly for the dodecahedron and the icosahedron. (This duality is the same as that used for planar maps in graph theory.)

Incidentally, the tetrahedron, cube and octahedron are the crystal structures of sodium sulphantimoniate, sodium chloride (common salt) and chrome alum, respectively.

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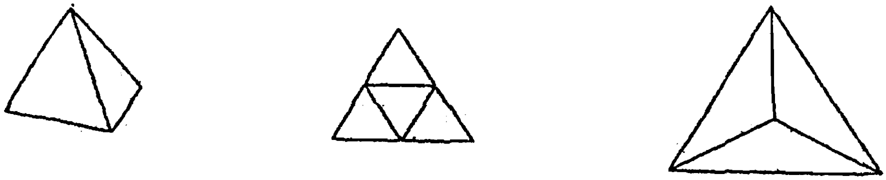
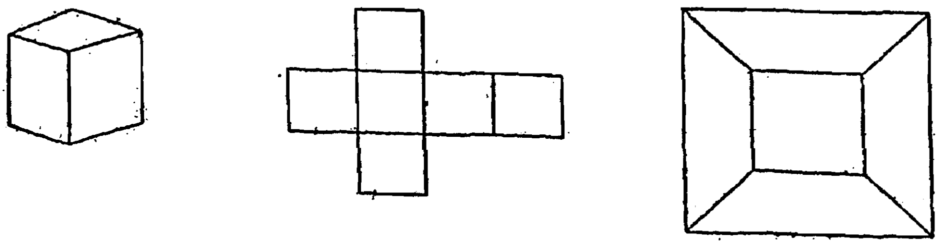
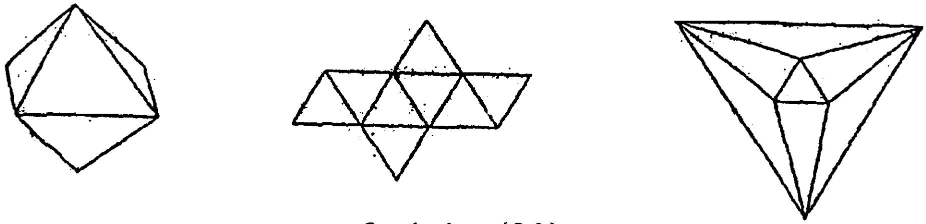
Box 1. Introduction to Geometry by H S M Coxeter, 1961.



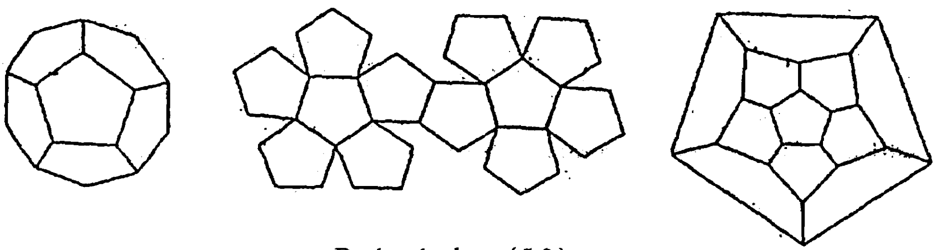
Tetrahedron {3,3}



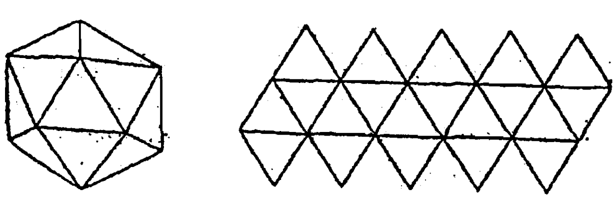
Cube {4,3}

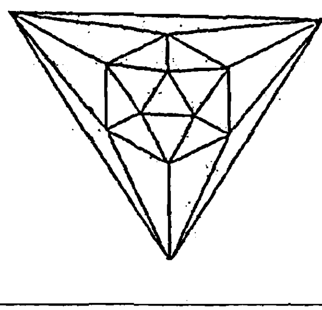


Octahedron {3,3}



Dodecahedron {5,3}





Icosahedron {3,3}

The skeletons of certain microscopic sea animals called *Circorrhegma dodecahedra* and *Circogonia icosahedra* (and some other viruses) are in the shape of a dodecahedron and an icosahedron, respectively, see Gallian (1999) and Coxeter (1948). In 1985, Robert Curl, Richard Smalley and Harold Kroto created a form of carbon by using a laser beam to vapourize graphite. The resulting molecule has 60 carbon atoms arranged in the shape of a football. Curl, Smalley and Kroto received the Nobel Prize for this discovery in 1996.

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We now show briefly how vertices, edges and faces, which we all understand intuitively, can be defined formally. An *extreme subset* of a convex set C is a convex set $D \subseteq C$ such that $C \in D, C \in AB, C \neq A, C \neq B$ and $A, B \in C \Rightarrow A, B \in D$.

Such an extreme subset is called a *vertex, edge* or *face* accordingly as it is of dimension 0, 1 or 2. The dimension of a non-empty proper subset of $\mathbb{R}^3$ is 0 if it is a singleton, 1 if it is contained in a line and is not a singleton and 2 if it is contained in a plane and is not contained in any line. Note that a cube has 8 vertices, 12 edges and 6 faces.

Recall the Krein–Milman theorem which was discussed in V S Sunder's article [3]. A simple consequence of the theorem is: *A convex polytope has finitely many vertices and is their convex hull.* Conversely, the convex hull of finitely many points is a convex polytope.

Every extreme subset of a convex polytope C is the intersection of C with a plane P such that C is contained in a half-space corresponding to P. An extreme subset of an extreme subset is an extreme subset. Each edge of C is the line segment joining two vertices of C and is on the boundary of exactly two faces. Each face of C is a convex polygon formed by some edges of C.

Since each face of a football is bounded, it can be proved

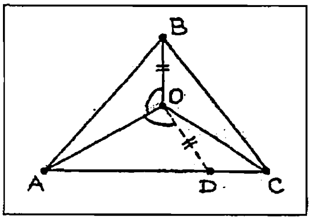


Figure 2.

that the football is bounded and, so, is a convex polytope. We omit this proof.

From now on, we consider only convex polytopes with dimension 3. Also, whenever we talk of $\angle ABC$, we shall mean that angle which is between 0° and 180° We prove the uniqueness of a football first assuming its existence and later prove the existence. We start with a simple result which is intuitively obvious.

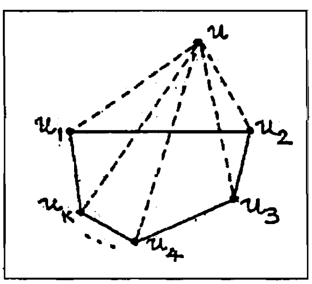
Lemma 1. There are at least 3 edges at every vertex of a convex polytope.

Lemma 2. (Euclid, XI.20) Suppose A, B, C and O are not coplanar. Then $\angle AOB + \angle BOC > \angle AOC$.

Proof: We may assume that $\angle AOC > \angle AOB$, for, otherwise the result is trivial. Let D be a point in the plane AOC such that $\angle AOD = \angle AOB$ and $OD = OB$. Refer to *Figure* 2. We may take C to lie on AD extended. Now triangles AOB and AOD are congruent, so $AB = AD$. Since $AB + BC > AC$, we get $BC > AC - AD = DC$. So comparing triangles DOC and BOC, we get $\angle BOC > \angle DOC$. So $\angle AOB + \angle BOC > \angle AOD + \angle DOC = \angle AOC$. □

Lemma 3. (Euclid, XI.21) The sum of the angles in all the faces at any vertex u of a convex polytope with dimension 3 is less than 360°.

Figure 3.



Proof: We may take the faces at u to be $u_i u u_{i+1}, i = 1, 2, \quad k$ where $u_1 u_2 \quad . u_k$ is a convex polygon in a plane P and $u \notin P$. See *Figure* 3. Let us call the angles of the type uu_iu_{i-1} or uu_iu_{i+1} *base angles*, angles of the type $u_{i-1}u_iu_{i+1}$ *polygonal angles* and angles of the type u_iuu_{i+1} *vertical angles.* Using the result that the sum of the angles in a triangle equals π, we see that the sum of the base angles and the vertical angles is $k\pi$. Using the fact that the sum of the two base angles at u_i is greater than the polygonal angle at u_i, we see that the sum of all the base angles is greater than the sum of all the polygonal angles which is $(k-2)\pi$. So the sum of all the vertical angles is less than 2π. This proves the lemma. □

At every vertex of a football, there is exactly one pentagonal face and so there are two hexagonal faces.

It may be worth noting here that the following simple 'proof' for the preceding lemma does not work always: let v be the foot of the perpendicular from u to the plane P. We may assume that v lies inside the convex polygon $u_1u_2 \; . u_k$. Then, it is perhaps natural to guess that angle u_iuu_{i+1} < angle u_ivu_{i+1} for each i, and so the lemma would follow. But, the inequality stated can be false if one of the angles vu_iu_{i+1} and $vu_{i+1}u_i$ is greater than a right angle (to get a counter-example, take angle vu_iu_{i+1} close to 180° and length uv moderately large).

Next, we can single out an observation about the football.

Theorem 1. At every vertex of a football, there are exactly three faces and so three edges.

Proof: Since the angles in a regular pentagon are 108° each and the angles in a regular hexagon are 120° each, there cannot be more than three faces at any vertex by lemma 3. So the theorem follows from lemma 1. □

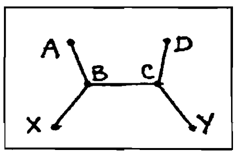
Lemma 4. If two regular polygons $. ABCD$ and $. XBCY$ in $\mathbb{R}^3$ have a common edge BC (see *Figure* 4), then $\angle ABX = \angle DCY$

Proof: We first clarify that a regular polygon is, by definition, planar. Now triangles ABX and DCY are congruent since each is the reflection of the other in the plane P perpendicularly bisecting BC. Thus, the lemma follows. □

Here is another observation about the football.

Theorem 2. At every vertex of a football, there is exactly one pentagonal face (and so there are two hexagonal faces).

Figure 4.



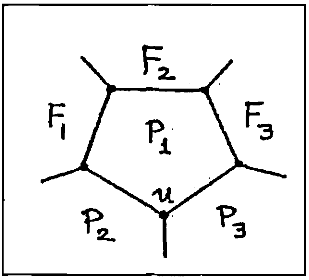


Figure 5.

Proof: Suppose that at a vertex u there are three pentagonal faces P_1, P_2 and P_3. See *Figure* 5. Then, lemma 4 applied to P_1 and P_2 gives F_1 is a pentagon. Similarly, F_2 and F_3 are pentagons. Since we can go from P_1 to any face by passing along adjacent faces, it follows that all faces are pentagons, a contradiction. (Incidentally, there is a convex polytope called dodecahedron with 12 faces all of which are regular pentagons.)

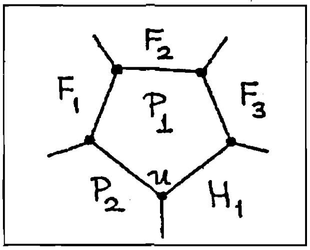


Figure 6.

Suppose next that at a vertex u there are two pentagonal faces P_1 and P_2 and a hexagonal face H_1 (see *Figure* 6). Then, lemma 4 applied to P_1 and F_1, gives F_2 is a pentagon. This gives a contradiction to lemma 4 when applied to P_1 and F_3.

Thus at any vertex there is at most one pentagonal face. By lemma 3, all the three faces at a vertex cannot be hexagonal, so the theorem follows. □

Lemma 5. Suppose $u_1u_2u_3u_4u_5$ is a regular pentagon in some plane in $\mathbb{R}^3$. Then there is a unique way in which two regular hexagons can be attached at u_1u_2 and u_2u_3 so that they have a common edge u_2v and lie above the plane of u_1u_2 . u_5. (See *Figure* 7).

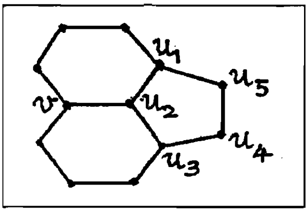
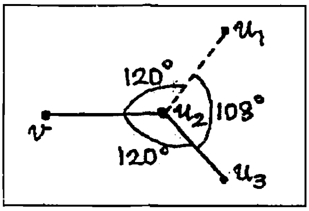


Figure 7.

Proof: Take $u_2 = (0,0,0), v = (-1,0,0)$ and $u_3 = (\alpha, \beta, 0)$. Refer to *Figure* 8. Since $\angle vu_2u_3 = 120°$ we have $-\alpha = \langle v, u_3 \rangle = \cos 120° = -1/2$. Here $\langle u, v \rangle$ denotes the dot product of u and v. Since $u_2u_3 = 1$, we may take $\beta = -\sqrt{3}/2$. Thus $u_3 = (1/2, -\sqrt{3}/2, 0)$.

Let $u_1 = (\gamma, \delta, \epsilon)$. Since $\angle vu_2u_1 = 120°$, we get $\gamma = 1/2$ as above. So $u_1 = (1/2, \delta, \epsilon)$ where $\delta^2 + \epsilon^2 = 3/4$. Now $\angle u_1u_2u_3 = 108°$ and $\cos 108° = (1 - \sqrt{5})/4$. So

Figure 8.



$$\frac{1 - \sqrt{5}}{4} = \langle u_1, u_3 \rangle = \frac{1}{4} - \frac{\sqrt{3}}{2}\delta.$$

Hence $\delta = \sqrt{5}/(2\sqrt{3})$ and $\epsilon = \pm 1/\sqrt{3}$. Assuming that u_1 lies above the x-y plane, we get $\epsilon = 1/\sqrt{3}$. Thus $u_1 = (1/2, \sqrt{5}/(2\sqrt{3}), 1/\sqrt{3})$. Since a regular polygon is determined by three consecutive vertices, it follows that the relative positions of the three polygons at u_2 are uniquely determined and the lemma follows. □

It is easy to write down the equations of the three planes at u_2 and so their normals at u_2. Using these, we can find the angle between the planes of the two hexagons to be $\cos^{-1}(\sqrt{5}/3) = \tan^{-1}(2/\sqrt{5}) \approx 41.81°$ and the angle between the planes of the pentagon and each of the hexagons to be $\tan^{-1}(3-\sqrt{5}) \approx 37.38°$.

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We can now prove that there is at most one football.

Theorem 3. Given a regular pentagon P in some plane in $\mathbb{R}^3$, a football with P as a face and lying on a given side of the plane of P is unique if it exists.

Proof: The positions of the five hexagons around P are unique by the preceding lemma. Imagine attaching regular pentagons and regular hexagons (with the side same as that of P) in the order shown in *Figure* 9. At each stage, the type of face to be used is unique by theorem 2 and its position is unique since three or four consecutive vertices of a regular polygon determine the polygon. Hence the uniqueness follows. □

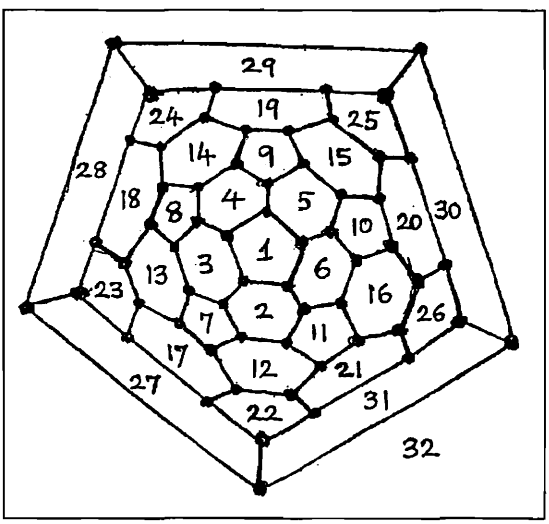

Figure 9.

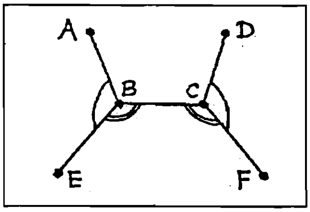

Figure 10.

Here is a result which provides the key to actually assembling the football thereby proving its existence.

Lemma 6. Let $ABCD$ be a regular polygon. Let BE and CF be such that $\angle ABE = \angle DCF$ and $\angle EBC = \angle FCB$ (see *Figure* 10). If E and F are both on the same side of the plane of $ABCD$, then E, B, C and F are coplanar.

Proof: We may take $B = (-1/2, 0, 0), C = (1/2, 0, 0)$ and $D = (a, b, 0)$. Then clearly $A = (-a, b, 0)$. Now let $F = (\alpha, \beta, \gamma)$ and $E = (\delta, \epsilon, \phi)$. We may also suppose that $BE = CF = BC$. Then

$$\cos \angle EBC = \langle (\delta + \frac{1}{2}, \epsilon, \phi), (1, 0, 0) \rangle = \delta + \frac{1}{2}$$

and $\cos \angle FCB = -(\alpha - \frac{1}{2})$. So $\delta = -\alpha$. Now $\cos \angle ABE = \cos \angle DCF$ gives $b\epsilon = b\beta$ and so $\epsilon = \beta$. Now $BE^2 = CF^2$ gives $\phi^2 = \gamma^2$. Since γ and ϕ have the same sign, they are equal. Thus $E = (-\alpha, \beta, \gamma)$ and E, B, C and F lie on the plane $\gamma y - \beta z = 0$. $\square$

We now prove that a football exists. Why, one may wonder, because all of us have seen footballs. Well, there is a problem here. Firstly, the footballs we see are not supposed to be footballs as defined here because nobody wants to play football with a solid with sharp edges and corners. (The edges of the football we see are geodesics and the faces are spherical regions.) Secondly it is possible that a football as defined here does not exist and the footballs we see are only approximations.

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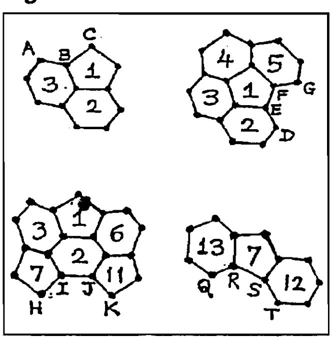
Theorem 4. The football referred to in theorem 3 exists.

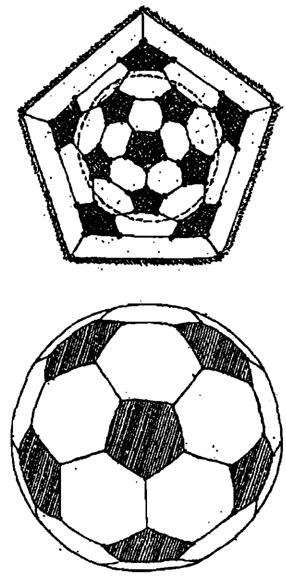
Proof: We show that the football can be assembled as shown in *Figure* 9 used in the proof of theorem 3. Refer also to *Figure* 11. We start with faces 1, 2 and 3. This is possible by lemma 5. By lemma 4, $\angle ABC = 120°$, so we can attach face 4 at ABC. Similarly we can attach face 5 also. Then by lemma 6, D, E, F and G are copla-

Figure 11.



nar, so face 6 can be fitted there. Then clearly faces 7 and through 11 can be fitted. Then, again by lemma 6, H, I, J and K are coplanar, so face 12 can be fitted there. Next we can fit faces 13 through 16. Now Q, R, S and T are coplanar, so face 17 can be fitted there. Proceeding thus we fit faces 18-21, then 22-26, then 27-31 and finally face 32. This proves that the football can be assembled and so exists. □

Box 2.



We next see how symmetric the football is. We start by showing that the vertices lie on a sphere. Note that the following analogue in two dimensions is false: the vertices of a convex polygon with all sides equal lie on a circle. The polygon shown in *Figure* 12 is far from equi-angular and can be perturbed further.

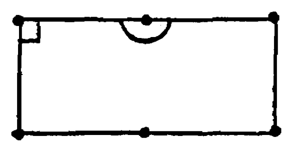
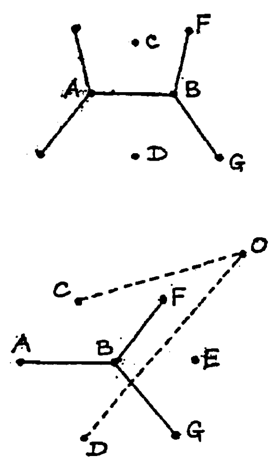


Figure 12.

Theorem 5. The normals to any three mutually adjacent faces of a football are concurrent at a point O which is the centre of a sphere on which the vertices lie.

Proof: By the normal to a face we mean the line passing through its centre and perpendicular to its plane. Refer to *Figure* 13. Let AB be the common edge between two faces and C and D the centres of the two faces. Then it is easy to see that the plane perpendicularly bisecting AB will contain the normals to the two faces. So these normals are coplanar. Since the faces are not parallel, these normals intersect at, say, O. Since O lies on the normal to face ABF, we have $OA = OB = OF$. Since O lies on the normal to face $ABG, OB = OG$. Thus $OF = OB = OG$. So O lies on the planes perpendicularly bisecting BF and BG. Hence O lies on the normal to the face FBG. Thus the normals to the three faces are concurrent at O and $OA = OB = OF = OG$. By proceeding through adjacent faces, we can see that O lies on the normal to every face. This proves the theorem. □

Figure 13.



It will be an interesting exercise to determine the radius of the sphere on which the vertices of the football lie, The numbers of vertices, edges and faces on the football can be counted from a drawing of the football but a bit of graph theory can be used too to find these.

given the length of an edge.

We now determine the numbers of vertices, edges and faces on the football. This will be needed later in part II where we determine its group of symmetries. Though these numbers can be counted from a drawing of the football, we will use a bit of graph theory to find these (partly explaining my interest in the topic).

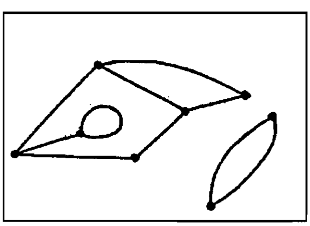
A (finite) *graph* G consists of a finite non-empty set V whose elements are called vertices and a finite collection E of unordered pairs (called edges) of elements of V A *plane graph* is a graph whose vertices are points in the plane and whose edges are arcs joining the vertices, no two of these arcs meeting each other except at the ends. A *face* of a plane graph G is a maximal connected region of the plane left when the vertices and edges of G are removed. It is easy to see that the vertex set of a graph G can be partitioned into its *components* such that we can go from every vertex in a component to every other vertex in the same component and to no vertex in any other component by travelling along edges (see *Figure* 14). We now prove a slight generalisation of a well-known formula so that we can use induction conveniently.

Lemma 7. (Euler's formula): For any plane graph G,

$$\nu - \epsilon + \gamma = 1 + p$$

where ν is the number of vertices, ϵ is the number of edges, γ is the number of faces (including the unbounded face) and p is the number of components.

Figure 14.



Proof: We prove the result by induction on ϵ. If $\epsilon = 0$, then $\gamma = 1$ and $p = \nu$, so the result follows. So assume the result for plane graphs with less than ϵ edges and let G have ϵ edges. If an edge belongs to a 'cycle', then by deleting this edge, we get a plane graph with ν vertices, $\epsilon - 1$ edges, $\gamma - 1$ faces and p components, so by induction hypothesis, we are done. If an edge uv does not belong to any cycle, then by deleting this edge, we get a plane graph with ν vertices, $\epsilon - 1$ edges, γ faces and $p + 1$ components, so by induction hypothesis, we are again done. □

Euler's polyhedral formula can be used to show that there are only five regular solids.

Theorem 6. A football has 60 vertices, 90 edges and 32 faces of which 12 are pentagons and 20 are hexagons.

Proof: Any football can be represented by its *Schlegel diagram* which is what the football (assumed to be transparent except for the edges) appears like when seen from a position just outside the centre of one face. This is like stereographic projection from the top (assumed to be not a vertex and not lying on any edge) of the sphere on which the vertices of the football lie, onto a horizontal plane below the sphere. The Schlegel diagram is a plane graph G, vertices, edges and faces of the football corresponding naturally to those of G, the face of the football nearest to the viewer corresponding to the unbounded face. Note that G has only one component, so Euler's formula reduces to $\nu - \epsilon + \gamma = 2$. By theorem 1, there are exactly three edges at every vertex and every edge is incident with exactly two vertices. Thus $2\epsilon = 3\nu$. Since every vertex is incident with exactly one pentagon and each pentagon is incident with exactly 5 vertices, it follows that the number of pentagons is $\nu/5$. Since every vertex is incident with exactly two hexagons and each hexagon is incident with exactly 6 vertices, it follows that the number of hexagons is $\nu/3$. Substituting these in Euler's formula we get

$$\nu - \frac{3\nu}{2} + \frac{\nu}{5} + \frac{\nu}{3} = 2,$$

so, $\nu = 60$. Now the theorem follows easily. □

Finally, we show how Euler's polyhedral formula can be used to show that there are only five regular solids. In a plane graph, a k-cycle refers to a sequence of edges of the type $v_1v_2, v_2v_3, \quad, v_{k-1}v_k, v_kv_1$ where the vertices $v_1, v_2, \quad, v_k$ are all distinct and $k \geq 1$.

Lemma 8. If each face of a connected plane graph G is a p-cycle for a fixed $p \geq 3$ and if there are $q \geq 3$ edges at every vertex of G, then $(p, q) = (3,3), (3,4), (3,5), (4,3)$ or $(5,3)$.

Proof: We first note that every edge joins two distinct vertices since if there is a self-loop, the face just inside it cannot be a p-cycle with $p \geq 3$. So, as in the proof of theorem 6, we get $q\nu = 2\epsilon = p\gamma$. Now, by Euler's formula, $\nu - \epsilon + \gamma = 2$. So

$$\frac{\nu}{\frac{1}{q}} = \frac{\epsilon}{\frac{1}{2}} = \frac{\gamma}{\frac{1}{p}} = \frac{\nu - \epsilon + \gamma}{\frac{1}{q} - \frac{1}{2} + \frac{1}{p}} = \frac{2}{\frac{1}{q} - \frac{1}{2} + \frac{1}{p}} = \frac{4pq}{2p - pq + 2q}$$

So $2p - pq + 2q > 0$ or $(p-2)(q-2) < 4$. It follows easily that $p \leq 5$. Moreover, if $p = 3$, then q can take only the values 3, 4 and 5. If $p = 4$ or 5, then q can take only the value 3. □

Lemma 9. (Euclid's Comment at the end of Book XIII): The Schlafli symbol of any regular solid (i.e., a 3-dimensional polytope with each face a regular p-gon and with exactly q faces at each vertex) is (3,3), (3,4), (3,5), (4,3) or (5,3).

Proof: We will give two proofs of this result, the first using Euler's formula. The Schlegel diagram of any regular solid with Schlafli symbol (p, q) is a plane graph G satisfying the hypothesis of lemma 8, so (p, q) can take only one of the five values mentioned in that lemma. This proves lemma 9.

We now give a second proof, essentially due to Euclid, which is applicable only to regular solids and which does not use Euler's formula. Since each angle in a regular p-gon is $(1-2/p)\pi$ and there are q such faces at any vertex, it follows from lemma 3 that $q(1-2/p)\pi < 2\pi$ which, on simplification, becomes exactly $2p - pq + 2q > 0$. Now the conclusion follows as in lemma 8. □

Now, if there is a regular solid with Schlafli symbol (p, q), then (p, q) can take only the five values stated in the preceding lemma. Using the equalities displayed in the proof of lemma 8 (or by direct counting), it is easy to find the numbers of vertices, edges and faces in each of the above five cases. Finally it can be shown, as for a football, that a regular solid with Schlafli symbol any one of the five referred to above, is unique. This shows that the platonic solids are the only 'regular solids' as stated by Euclid at the end of Book XIII.

Incidentally, a plane *tessellation* is a covering of the plane with nonoverlapping (except for the edges between) polygons. A tessellation is a *regular tessellation* if the polygons are all regular p-gons for some p. If a regular plane tessellation exists with p-gons and if there are q such polygons at *some* vertex, then we have $q(1 - 2/p)\pi = 2\pi$, so $(p-2)(q-2) = 4$. It follows that $(p,q) = (3,6), (4,4)$ or $(6,3)$. Each of these is actually possible as the tessellations in *Figure* 15 show. (Note that, now, p determines q and the same number of polygons occurs at every vertex; this was not assumed in the definition).

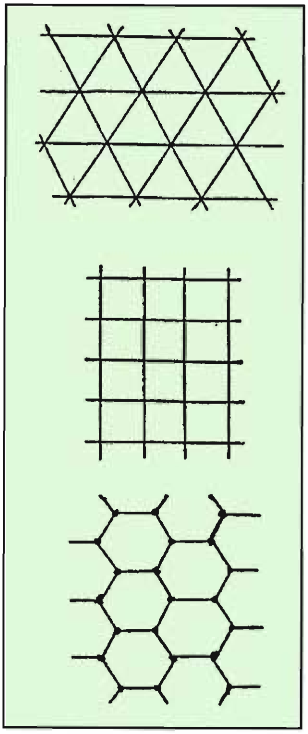


Figure 15.

We mention in passing that a beehive looks quite like a 3-dimensional convex polytope with every face a regular hexagon and with three faces meeting at every vertex. However, this cannot really be, since the sum of the angles at any vertex will then be 360° and the polytope has to be planar by lemma 3. Thus the polytope with the stated properties does not exist and a beehive is only a clever approximation. This shows the need for proving the existence of a football.

We end the first part here. In the next part, we shall identify the group of symmetries of a football. That will also contain a discussion on the groups of symmetries of some other objects in 3-space.

Suggested Reading

[1] H S M Coxeter, *Regular Polytopes*, Methuen, 1948.

[2] J A Gallian, *Contemporary Abstract Algebra*, 4th Ed., Narosa, 1999.

[3] V S Sunder, *Some aspects of convexity* I & II, Vol.5, No. 5, pp.5-16; No.6, pp.49-59, 2000.

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