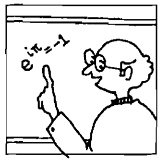

In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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How Safe is Sam Lloyd's Bet? The 15-Puzzle and Beyond

A Puzzle

Look at the picture here of a 4×4 square on which 15 coins have been placed leaving the last square empty.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Sam Lloyd, the originator of this puzzle offered in 1879 a prize of $1000 to anyone who could get the arrangement with 14 and 15 switched above.

The idea is to slide the coins utilising the empty square and to find out what kind of arrangements are possible. As the story goes, Sam Lloyd, the originator of this puzzle offered in 1879 a prize of $1000 to anyone who could get the arrangement with 14 and 15 switched above. He knew his money would be safe with him. We hope to convince the reader, of this and much more by the end of this discussion.

To get a feeling for this the natural thing is to first look at the simplest analogue of it viz. the figure

1	2
3	

A moment's thought would convince the reader that the pattern

1	3
2	

can never occur. In fact, only those arrangements are possible where $1, 2, 3$ occur in that order (if we go clockwise).

Now, let us see what happens if we look at:

1	2	3
4	5	6
7	8	

A little bit of fiddling with it already shows that many arrangements are possible. However, it is not easy to get even a candidate for an arrangement which might not be possible. Having said that, it is perhaps justified to introduce some technology which might allow us to analyse the puzzle systematically. This technology is in the form of what are known as permutation groups.

Permutation Groups

A permutation of a finite set $\{a_1, a_2, \quad, a_n\}$ of objects is, as evident from the normal meaning of the English word, a correspondence which associates to each member of the set a unique member. For instance, the correspondence that associates to each member the member itself, is called the identity permutation. It is at once clear that so far as a permutation is concerned the elements $a_1, a_2, \quad, a_n$ are just certain symbols and what is important is only which symbol corresponds to which, under the permutation. In other words, one might think of the permutation as a correspondence of the *subscripts.* So it is customary to write $\{1, 2, \quad, n\}$ for the set instead of $\{a_1, a_2, \quad, a_n\}$.

Notice that common sense tells us that any permutation is a product of transpositions – after all we mortals have only two hands!

Now, evidently one can define the *composition* of two permutations as the process of applying one after the other. Of course, a momentary thought makes it clear that the order in which they are applied is important and the final effect may vary according to the order of application. Under this *product* operation the set S_n of all permutations of n objects has the structure of a group (see [1], p.50 for a brief discussion of this notion).

It is easy to see that there are exactly $n!$ permutations in S_n. The following notation turns out to be convenient while describing a permutation. To motivate this, look at $1, 2, 3$ and the permutation σ which sends 1 to 2, 2 to 3 and 3 to 1. One writes σ = (1,2,3). If τ is the permutation which sends 1 to 2, 2 to 1 and 3 to itself, one writes $\tau = (1, 2)(3)$ or simply $(1, 2)$. In general, $\sigma = (i_1, i_2, \quad , i_r)$ in S_n means that $i_1 \to i_2 \to \quad \to i_r \to i_1$ and $\sigma(i) = i$ for all $i \neq i_1, i_2, \quad , i_r$. This is called an *r-cycle.* A 2-cycle is also called a *transposition* for obvious reasons. Let us use the convention that $\sigma\tau$ denotes the permutation where σ is applied after τ. For instance, in S_3, look at σ = (1,2) τ = (2,3). Then $\sigma\tau$ = (1,2,3).

Exercises

(i) *Let $\sigma \in S_n$ be an r-cycle. Then, show that σ has order r i.e. σ^r is the identity permutation and it is the least positive power when this happens.*

(ii) *Prove that disjoint cycles commute.*

(iii) *Show that any $\sigma \in S_n$ is a product of disjoint cycles in a unique manner, apart from the order.*

Notice that common sense tells us that any permutation is a product of transpositions – after all we mortals have only two hands! Let us be warned that this expression as a product of transpositions is far from unique. Witness $(1, 2)(2, 3) = (2, 3)(1, 3) = (1, 2)(1, 3)(1, 2)(1, 3)$. The astute reader, however, notices that the number of transpositions occurring are even.

It is easily verified that: *A permutation cannot be a product of an even number of transpositions on the one hand and of an odd number of transpositions on the other.*

'Never odd or even' is a palindrome.

Thus, one naturally has the notions of *even permutations* and of *odd permutations.* As identity is an even permutation and any two even numbers add to give an even number, the set A_n of all even permutations is a subgroup of S_n i.e., a subset of S_n which is a group in its own right. A_n is called the alternating group.

Here are two very useful observations:

(i) A *conjugate* of an r-cycle is again an r-cycle viz.,

$$\sigma(i_1, i_2, \quad , i_r)\sigma^{-1} = (\sigma(i_1), \sigma(i_2), \quad , \sigma(i_r))$$

This is evident. In fact, *any two $r-$cycles are conjugates*!

For, if $(i_1, i_2, \quad , i_r)$ and $(j_1, j_2, \quad , j_r)$ are r-cycles, look at a permutation σ which sends each i_k to j_k and $\{1, 2, \quad n\} \setminus \{i_1, i_2, \quad , i_r\}$ to $\{1, 2, \quad , n\} \setminus \{j_1, j_2, \quad , j_r\}$ in a bijective fashion.

(ii)For any $n \geq 3, A_n$ is *generated by* 3-cycles i.e. even permutations are expressible as products of 3-cycles.

Proof: Evidently every 3-cycle (a, b, c) is $(a, b)(b, c)$, an even permutation. Further, any element of A_n is a product of an even number of transpositions. So it is enough to show the product of two transpositions is equal to a product of 3-cycles. For this, we observe:

$$(a, b)(c, d) = (a, b, c)(b, c, d)$$

$$(a, b)(a, c) = (a, c, b)$$

$$(a, b)(a, b) = \mathrm{Id} = (a, b, c)^3$$

Even permutations are expressible as products of 3-cycles.

With these tools in hand, let us take a new look at the puzzle. In fact, let us look at the general $n \times n$ puzzle where the last location is empty.

The good positions correspond to even permutations. This explains Sam Lloyd's confidence – (14, 15) is an odd permutation!

(n^2-1)- Puzzle

Use the numbers $1, 2, \quad , n^2$ to indicate locations in the frame which holds the squares. In the starting position, square 1 is in location 1, and the blank space in the location n^2.

1	2			$n-1$	n
$n+1$	$n+2$			$2n-1$	$2n$
n^2-n+1	n^2-n+2			n^2-1	

The location number remains fixed; the numbered squares in the locations may change. A permutation $\sigma \in S_{n^2}$ can be viewed as permuting any given arrangement A as follows. The square in location i of A is moved by σ to the location $\sigma(i)$ in the new arrangement $\sigma(A)$. For instance, the transposition $(15, 16)$ applied to the starting position, has blank space moved to 15$^{\text{th}}$ location in the new arrangement in the 15-puzzle.

Our aim would be to say something about the *good positions* i.e. the arrangements which can be obtained from the starting position by sliding.

We start with the following trivial observation. Given any arrangement, we can perform a series of simple moves in order to bring the blank space to the n^2-th location. Let H be the subset of S_{n^2} consisting of all elements that correspond to arrangements which are obtained from the starting position by a series of moves which ends with the blank square in the location n^2. Then, by its very definition, H is the subset consisting of all those permutations which give all the *good positions.* We shall show now that H is a group and that it consists entirely of even permutations. This would furnish the explanation for Sam Lloyd's confidence – $(14, 15)$ is an odd permutation!.

Claim : *H is a subgroup of A_{n^2-1}.*

Proof: Every simple move corresponds to a transposition (a, b) where either a or b is the location of the blank square before the move is applied. Suppose $\sigma \in H$ Then, as σ leaves any arrangement with the n^2-th location blank to an arrangement with the same property, we can write

$$\sigma = (n^2, x_{t-1})(x_{t-1}, x_{t-2}) \quad (x_2, x_1)(x_1, n^2)$$

It is clear that H is a subgroup as the product of any two elements of this form is again of this form. Let $\sigma \in H$ be a product of t transpositions. While operating by σ, let u, d, l, r denote number of moves up, down, left and right, respectively, of the blank square. Therefore, $u + d + l + r = t$. As the blank square returns to the original position, one must have $l = r \quad u = d$. Thus $t = 2(l + u)$ i.e., t is even and $\sigma \in A_{n^2}$. Note that $\sigma \in S_{n^2-1}$ as the last square stays empty even after σ is applied; hence $\sigma \in A_{n^2-1}$. In other words, we have established that $H \subset A_{n^2-1}$.

Let us stop for a moment to notice what H is in the simplest case viz., that of $n = 2$. We see that

$$H = \{Id, (1,3,2), (1,2,3)\} \text{as}$$

1	2
3	

3	1
2	

2	3
1	

are the good positions with the last place blank. In other words, $H = A_3$. Now we go on to show that H is actually the whole of A_{n^2-1} even in general. Let's first look at the original 15-puzzle i.e., the case $n = 4$.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Let us look at the following permutations.

$$\alpha = (1, 2, 3, 4, 8, 12, 15, 14, 13, 9, 5)$$

$$\begin{aligned}\beta &= (6,7,8,12,15,14,10)\\ \gamma &= (11,12,15)\\ \sigma &= (2,3,4,8,12,15,14,10,6)\end{aligned}$$

The reader can see that these are all in H Notice a pattern among the elements above, which is in terms of the portions of the square these elements leave invariant.

As $\beta^5 = (6,14,12,7,10,15,8), \gamma = (11,12,15) \in H$ we have

$$\beta^5\gamma\beta^{-5} = (\beta^5(11), \beta^5(12), \beta^5(15)) = (11,7,8) =$$

$$(7,8,11) \in H.$$

Also, if τ is a power of α or of σ, then

$$\tau(7,8,11)\tau^{-1} = (7,\tau(8),11).$$

Now we observe that every $x \neq 7, 11$ can be written as $\alpha^i(8)$ or $\sigma^j(8)$ which implies that $(7,x,11) = (7,\tau(8),11)$ $= \tau(7,8,11)\tau^{-1} \in$ H (where τ is a power of α or σ). Thus $(7,x,11)(7,y,11)(7,x,11)^{-1} = (x,y,7) \in H$ Hence $(7,z,11)(x,y,7)(7,z,11)^{-1} = (x,y,z) \in H$ Thus H contains all the 3-cycles.

It turns out that exactly the same proof can be modified to analyse the $n \times n$ array for any $n \geq 4$. Note that if $n = 3$, the analogues of α and β both coincide and thus, the same proof will not carry over. However the result is true (see *Box* 1). This is an aspect which frequently occurs in mathematics. Some proofs require enough dimensions or – to put it in more colourful language – more elbow-room!

1	2			$n-1$	n
$n+1$	$n+2$		$\cdot$	$2n-1$	$2n$
n^2-2n+1	n^2-2n+2			n^2-n-1	n^2-n
n^2-n+1	n^2-n+2			n^2-1	

Box 1.

1	2	3
4	5	6
7	8	

We look at the 3×3 puzzle above and consider the cycles

$\alpha = (1,2,3,6,8,7,4)$, $\beta = (4,5,6,8,7)$, $\gamma = (2,3,6,8,5)$, $\delta = (5,6,8)$.

If we show that all 3-cycles in which 5 occurs, are in H, it would follow that $H = A_8$ as $(x,y,z) = (z,x,5)(z,y,5)(5,x,z)$. This is seen as follows.

$$
\begin{array}{lcl}
(5,6,8) & = & \delta \\
(5,8,7) & = & \alpha\delta\alpha^{-1} \\
(5,7,4) & = & \alpha^2\delta\alpha^{-2} \\
(5,4,1) & = & \alpha^3\delta\alpha^{-3} \\
(5,3,6) & = & \alpha^{-1}\delta\alpha \\
(5,2,3) & = & \alpha^{-2}\delta\alpha^2 \\
(5,1,2) & = & \alpha^{-3}\delta\alpha^3 \\
(6,5,7) & = & \delta\alpha\delta\alpha^{-1}\delta^{-1} \\
(8,5,4) & = & (5,8,7)(5,7,4)(5,8,7)^{-1} \\
(5,6,4) & = & \delta(8,5,4)\delta^{-1} \\
(7,5,1) & = & (5,7,4)(5,4,1)(5,7,4)^{-1} \\
(4,5,2) & = & (5,4,1)(5,1,2)(5,4,1)^{-1} \\
(1,5,3) & = & (5,1,2)(5,2,3)(5,1,2)^{-1} \\
(2,5,6) & = & (5,2,3)(5,3,6)(5,2,3)^{-1} \\
(3,5,8) & = & (5,3,6)\delta(5,3,6)^{-1} \\
(5,3,7) & = & (5,3,6)(6,5,7)(5,3,6)^{-1} \\
(3,5,4) & = & (5,3,6)(5,6,4)(5,3,6)^{-1} \\
(2,7,5) & = & (6,5,7)(2,5,6)(6,5,7)^{-1} \\
(8,2,5) & = & (4,5,2)(8,5,4)(4,5,2)^{-1} \\
(6,1,5) & = & (2,5,6)(5,1,2)(2,5,6)^{-1} \\
(1,8,5) & = & (3,5,8)(1,5,3)(3,5,8)^{-1}
\end{array}
$$

There are $\binom{7}{2}$ (twenty one) 3-cycles containing 5. Therefore we have listed all of them.

Given a position, one can decide whether it is good or not by writing it as a product of transpositions. Once it is known to be good, we can write it as a product of 3-cycles and go through the above analysis to actually arrive at a sequence of sliding moves which reaches the starting position.

Look at the cycles

$$\sigma_1 = (1, 2, \quad , n, 2n, \quad , n^2 - n, n^2 - 1, \quad , n^2 - n + 1, \quad , n + 1)$$

$$\sigma_2 = (n + 2, n + 3, \quad 2n, \quad , n^2 - n, n^2 - 1, \quad , n^2 - n + 2, \quad 2n + 2)$$

$$\sigma_{n-2} = (n^2 - 2n - 2, n^2 - 2n - 1, n^2 - 2n, n^2 - n, n^2 - 1, n^2 - 2, n^2 - n - 2) = \beta$$

$$\sigma_{n-1} = (n^2 - n - 1, n^2 - n, n^2 - 1) = \gamma$$

$$\sigma_n = (n^2 - 3n - 2, ., n^2 - 3n, ., n^2 - n, n^2 - 1, n^2 - 2, ., n^2 - 2n - 2) = \sigma.$$

Clearly $\sigma_1, \quad , \sigma_n \in H$ In this case all the elements $\sigma_1, \quad , \sigma_{n-3}$ together play the role that α plays in the 15-puzzle. Now

$$\beta^5 = (n^2-2n-2, n^2-2, n^2-n, n^2-2n-1, n^2-n-2, n^2-1, n^2 - 2n)$$

which gives us

$$\begin{aligned} \beta^5\gamma\beta^{-5} &= (\beta^5(n^2 - n - 1), \beta^5(n^2 - n), \beta^5(n^2 - 1)) \\ &= (n^2 - n - 1, n^2 - 2n - 1, n^2 - 2n) \\ &= (n^2 - 2n - 1, n^2 - 2n, n^2 - n - 1) \in H. \end{aligned}$$

If τ is any power of $\sigma_1, \sigma_2, \quad , \sigma_{n-3}$ or σ_n, then

$$\tau(n^2 - 2n - 1, n^2 - 2n, n^2 - n - 1)\tau^{-1} = (n^2 - 2n - 1, \tau(n^2 - 2n), n^2 - n - 1)$$

as τ moves neither $n^2 - n - 1$ nor $n^2 - 2n - 1$. Moreover, every $x \neq n^2 - n - 1, n^2 - 2n - 1$ has the form $\tau^i(n^2 - 2n)$ for some i, where τ is one of $\sigma_1, \sigma_2, \quad , \sigma_{n-3}$

or σ_n which gives $(n^2-2n-1, x, n^2-n-1) \in H$ for all x. So,

$$(n^2-2n-1, x, n^2-n-1)(n^2-2n-1, y, n^2-n-1)(n^2-2n-1, x, n^2-n-1)^{-1} = (x, y, n^2-2n-1) \in H.$$

Thus,

$$(n^2-2n-1, z, n^2-n-1)(x, y, n^2-2n-1)(n^2-2n-1, z, n^2-n-1)^{-1} = (x, y, z) \in H.$$

In other words, for any n, the good positions are exactly those which can be reached by applying any permutation in A_{n^2-1} to the starting position.

To conclude our discussion, it is clear from the above analysis that given a position, one can decide whether it is good or not by writing it as a product of transpositions. Once it is known to be good, we can write it as a product of 3-cycles and go through the above analysis to actually arrive at a sequence of sliding moves which reaches the starting position. However, this may not be the quickest way. Hopefully, the reader is convinced now that one knows much more about this puzzle than the mere fact that Sam Lloyd's money was safe!

Given a position, one can decide whether it is good or not by writing it as a product of transpositions. Once it is known to be good, we can write it as a product of 3-cycles to actually arrive at a sequence of sliding moves which reaches the starting position.

Finally, we draw attention to the fact that the 15-puzzle admits of a graph-theoretic interpretation and may be generalised in that direction (See [2]). We recommend [3] as a source for a nice discussion of the 15-puzzle.

Suggested Reading

[1] B Sury, An introduction to Galois theory, *Resonance*, Vol. 4, No. 10, pp.47-60, 1999.

[2] Aaron F Archer, A Modern Treatment of the 15-Puzzle, *American Mathematical Monthly*, Vol.106, pp. 793-799, 1999.

[3] N McCoy and G J Janusz, *Introduction to Modern Algebra*, Universal Book Stall, New Delhi, 4th ed., 1990.