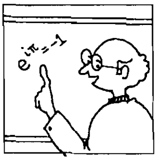
Classroom



In this section of* Resonance, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

On IBM's Millennial Puzzle

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We give lower and upper bounds on the number of n-digit numbers that are multiples of 2^n and which have no zeros in their decimal representation.

Introduction

Puzzle enthusiasts may like to visit the IBM website at *http: //www.research.ibm.com /features/ ponder/* where problems are posted every month and readers are invited to submit their solutions. Provided the submitted solution is correct, the reader also gets to have her/his name appear among the list of solvers. The problem for December 1999 reads as

> Consider the integer $N = 2^{1999}$. Is there a positive integer multiple of N whose decimal representation does not contain the digit 0? How would you construct such an integer or prove that it doesn't exist?

In this article we show, more generally, that for any n there are at least 4^n n-digit numbers that are multiples of 2^n and have no zeros in their decimal representation. We then show how we can go about constructing such a number using just 2 digits, say 1 and 2. Next we improve the lower bound of 4^n and give an upper bound as well.

The Lower Bound

We fix some notations. Let $\mathcal{S}_n$ denote the set of all n-digit numbers that have no zeros in their decimal representation (there are 9^n of them!). Let $\mathcal{E}_n$ and $\mathcal{O}_n$ be the set of *even* and *odd* multiples of 2^n in $\mathcal{S}_n$. Let $E_n = |\mathcal{E}_n|$ and $O_n = |\mathcal{O}_n|$ be the sizes of $\mathcal{E}_n$ and $\mathcal{O}_n$, respectively. Let $T_n = E_n + O_n$ be the total number of multiples of 2^n in $\mathcal{S}_n$.

Our first goal is to show, by induction on n, that $T_n \geq 4^n$. Clearly, $T_1 = 4 = 4^1$ since the single digit numbers that are multiples of 2 are $2, 4, 6$ and 8. Assume the result to be true for $n = k$ $(k \geq 1)$.

Now let $A = A_0A_1 \ldots A_k$ be a $(k+1)$-digit number which is a multiple of 2^{k+1} and the digits $A_i \neq 0$ for all i. Then $A = 10^k A_0 + Y$ where $Y = A_1 \ldots A_k$. If A_0 is odd (resp. even) then $10^k A_0$ is an odd (resp. even) multiple of 2^k. Then for 2^{k+1} to divide A, Y must be an odd (resp. even) multiple of 2^k.

Hence the number of multiples of 2^{k+1} with A_0 odd (resp. even) is $5\,O_k$ (resp. $4\,E_k$) because there are 5 possible choices for A_0 namely $1, 3, 5, 7$ and 9 (resp. 4 choices namely $2, 4, 6$ and 8).

Thus we see that

$$
\begin{aligned}
T_{k+1} &= 5O_k + 4\,E_k = 4T_k + O_k && (1)\\
&\geq 4T_k \geq 4 * 4^k \text{ (by induction hypothesis)} = 4^{k+1}
\end{aligned}
$$

proving our result.

Returning to the IBM puzzle, to construct a (1999-digit) number without zeros divisible by 2^{1999}, we need only 2 digits, one odd and one even, say 1 and 2. After constructing the number $A_k A_{k-1} \ldots A_1$, if the quotient on dividing this by 2^k is odd, set $A_{k+1} = 1$, otherwise set $A_{k+1} = 2$. Then the number $A_{k+1} A_k \ldots A_1$ is divisible by 2^{k+1}.

Thus our sequence of numbers would be

2, 12, 112, 2112, 22112, 122112, 2122112, ...

Improving the Bounds

The bound we have so far for T_n is

$$4^n \leq T_n \leq 5^n$$

The upper bound follows from the fact that there are 5^n multiples of 2^n that have n digits or less. Equation 1 shows that, in order to improve the bounds, we need to get good estimates of E_n and O_n. From *Table* 1, we see that E_n and O_n are nearly equal even for large values of n.

Table 1.

n	E_n	O_n	$T_n = E_n + O_n$	$T_n^{1/n}$
1	2	2	4	4.00
2	9	9	18	4.24
3	41	40	81	4.33
4	182	182	364	4.37
5	819	819	1638	4.39
6	3685	3686	7371	4.41
7	16582	16588	33170	4.42
8	74639	74629	149268	4.43
9	335852	335849	671701	4.44
10	1511320	1511333	3022653	4.45
11	6800982	6800963	13601945	4.45
12	30604369	30604374	61208743	4.46

To estimate E_n and O_n, let us consider $n = 3$. The sets $\mathcal{E}_3$ and $\mathcal{O}_3$ are

$$\begin{aligned}\mathcal{E}_3 &= \{112, 128, 144, 176, 192, 224, 256, 272, 288,\\ &\quad 336, 352, 368, 384, 416, 432, 448, 464, 496,\\ &\quad 512, 528, 544, 576, 592, 624, 656, 672, 688,\\ &\quad 736, 752, 768, 784, 816, 832, 848, 864, 896,\\ &\quad 912, 928, 944, 976, 992\}\\ \mathcal{O}_3 &= \{136, 152, 168, 184, 216, 232, 248, 264, 296,\\ &\quad 312, 328, 344, 376, 392, 424, 456, 472, 488,\\ &\quad 536, 552, 568, 584, 616, 632, 648, 664, 696,\\ &\quad 712, 728, 744, 776, 792, 824, 856, 872, 888,\\ &\quad 936, 952, 968, 984\}\end{aligned}$$

We note that adding 200 to a number in $\mathcal{E}_3$ moves it either outside S_3 or to $\mathcal{O}_3$ and vice versa; for example the numbers 128, 328, 528, 728 and 928 are alternately in the sets $\mathcal{E}_3$ and $\mathcal{O}_3$. This generalises to the following observation:

$$\text{A number } x \text{ is an even multiple of } 2^n \Leftrightarrow \\ x \pm 2 * 10^{n-1} \text{ is an odd multiple of } 2^n$$

Let $\ddot{E}_n$ (resp. $\ddot{O}_n$) be the number of elements in $\mathcal{E}_n$ (resp. $\mathcal{O}_n$) between 10^{n-1} and $3 * 10^{n-1}$. Let $\dot{E}_n$ (resp. $\dot{O}_n$) be the number of elements in $\mathcal{E}_n$ (resp. $\mathcal{O}_n$) between 10^{n-1} and $2 * 10^{n-1}$.

Then, from the above observation,

$$E_n = \ddot{E}_n + \ddot{O}_n + \ddot{E}_n + \ddot{O}_n + \dot{E}_n = 2(\ddot{E}_n + \ddot{O}_n) + \dot{E}_n \quad (2)$$

$$O_n = \ddot{O}_n + \ddot{E}_n + \ddot{O}_n + \ddot{E}_n + \dot{O}_n = 2(\ddot{E}_n + \ddot{O}_n) + \dot{O}_n. \quad (3)$$

Now we observe that removing the most significant digit in each number in the first rows of $\mathcal{E}_3$ and $\mathcal{O}_3$ gives us all the 2-digit multiples of 4, namely $12, 28, 44,\quad 96$. For numbers between 100 and 200, the result is an *odd* multiple of 4 while for numbers between 200 and 300, the result is an *even* multiple of 4.

These observations in fact generalise for any n, namely a number $x \in \mathcal{O}_{n-1}$ if and only if the n-digit number $1\,x$ formed by adding the digit 1 to the front of x is a multiple of 2^n. For an even multiple, we add the digit 2 to the front. This is encapsulated by the following two equations.

$$\begin{aligned} \ddot{E}_n + \ddot{O}_n &= T_{n-1} && (4) \\ \dot{E}_n + \dot{O}_n &= O_{n-1}. && (5) \end{aligned}$$

These equations can now be used to prove the following result.

Theorem 1. *Let*

$$\alpha = (\sqrt{17} - 3)/2$$

be one solution to the quadratic equation $(2+x)/(4+x) = x$. *Then,*

$$\frac{2}{4+\alpha} T_n \leq O_n \leq \alpha T_n. \tag{6}$$

Proof: We prove this by induction on n. The case $n = 1$, is easily verified as $O_1 = 0.5T_1$. From (4) and (5) and the induction hypothesis, we get $\dot{O}_n \leq O_{n-1} \leq \alpha T_{n-1} = \alpha(\ddot{E}_n + \ddot{O}_n)$. Also by (2), $E_n \geq 2(\ddot{E}_n + \ddot{O}_n)$. Hence, by (3)

$$O_n \leq (2+\alpha)(\ddot{E}_n + \ddot{O}_n) \leq \frac{2+\alpha}{2} E_n = \frac{2+\alpha}{2}(T_n - O_n).$$

Simplifying,

$$O_n \leq \frac{2+\alpha}{4+\alpha} T_n = \alpha T_n.$$

To get the lower bound, we show similarly that

$$T_n - O_n = E_n \leq (2+\alpha)(\ddot{E}_n + \ddot{O}_n) \leq \frac{2+\alpha}{2} O_n$$

from which we get

$$O_n \geq \frac{2}{4+\alpha} T_n.$$

This proves the result. □

From (1) and (6) and using α's value, we get

$$4.438\, T_n \leq T_{n+1} \leq 4.56\, T_n.$$

From the table, we see that $T_9 \geq (4.438)^9$ and hence for $n > 8$,

$$(4.438)^n \leq T_n \leq (4.56)^n \tag{7}$$

where the values are given to 3 decimals.

Further Improvements

We now provide a more careful approximation of the terms $\dot{E}_n$ and $\dot{O}_n$ in (2) and (3) thus giving us improved lower and upper bounds for T_n.

From (5) we see that $\dot{E}_n$ and $\dot{O}_n$ together count the total number of odd multiples O_{n-1}. To see what they represent individually, we go back to the example on page 84. Here $\dot{E}_3 = 5$ counts the number of elements in the set $\{112, 128, 144, 176, 192\}$. Removing the most significant digit leaves us with the set $\{12, 28, 44, 76, 92\}$. All the numbers in this set are multiples of 4 with the quotient congruent to (3 mod 4). In the case of $\dot{O}_3$, the corresponding multiples of 4 have quotient congruent to (1 mod 4). This naturally generalises for any n as follows.

Define $\mathcal{O}_{n,1}$ and $\mathcal{O}_{n,3}$ to be the set of elements in $\mathcal{O}_n$ that are multiples of 2^n with the quotient congruent to (1 mod 4) and (3 mod 4), respectively. Let $O_{n,1} = |\mathcal{O}_{n,1}|$ and $O_{n,3} = |\mathcal{O}_{n,3}|$. Then we have

$$\dot{O}_n = O_{n-1,1} \quad \dot{E}_n = O_{n-1,3}. \tag{8}$$

This follows because a number $x \in \mathcal{O}_{n-1,1}$ if and only if the n-digit number $1\,x$ is an odd multiple of 2^n. When $x \in \mathcal{O}_{n-1,3}$, then $1\,x$ is an even multiple of 2^n.

Thus by (8), to estimate $\dot{O}_n$ and $\dot{E}_n$, it suffices to find bounds for $O_{n-1,1}$ and $O_{n-1,3}$. To do this, let us go back to our example and write down the sets $\mathcal{O}_{3,1}$ and $\mathcal{O}_{3,3}$ which are

$$\begin{aligned}\mathcal{O}_{3,1} &= \{136, 168, 232, 264, 296, 328, 392, 424, 456, 488,\\ &\quad 552, 584, 616, 648, 712, 744, 776, 872,\\ &\quad 936, 968\}\\ \mathcal{O}_{3,3} &= \{152, 184, 216, 248, 312, 344, 376, 472,\\ &\quad 536, 568, 632, 664, 696, 728, 792, 824, 856, 888,\\ &\quad 952, 984\}\end{aligned}$$

We note that adding 400 to a number moves it from $\mathcal{O}_{3,1}$ either outside S_3 or to $\mathcal{O}_{3,3}$ and vice versa; for example, the numbers 136, 536 and 936 are alternately in the sets $\mathcal{O}_{3,1}$ and $\mathcal{O}_{3,3}$. This generalises to the following observation:

For a number $x \in \mathcal{O}_n$, $x \in \mathcal{O}_{n,1} \Leftrightarrow x + 4 * 10^{n-1} \in \mathcal{O}_{n,3}$ or $x - 4 * 10^{n-1} \in \mathcal{O}_{n,3}$.

Hence we split the interval $[10^{n-1}, 10^n]$ into subintervals $[10^{n-1}, 5*10^{n-1}]$, $[5*10^{n-1}, 9*10^{n-1}]$ and $[9*10^{n-1}, 10^n]$ and use the above observation to write

$$O_{n,1} = \ddot{O}_{n,1} + \ddot{O}_{n,3} + \dot{O}_{n,1} \tag{9}$$
$$O_{n,3} = \ddot{O}_{n,3} + \ddot{O}_{n,1} + \dot{O}_{n,3}. \tag{10}$$

Here $\ddot{O}_{n,1}$ and $\dot{O}_{n,1}$ (resp. $\ddot{O}_{n,3}$ and $\dot{O}_{n,3}$) are the numbers of elements in $\mathcal{O}_{n,1}$ (resp. $\mathcal{O}_{n,3}$) in the intervals $[10^{n-1}, 5 * 10^{n-1}]$ and $[10^{n-1}, 2 * 10^{n-1}]$ respectively.

Clearly, $\dot{O}_{n,1} \leq \ddot{O}_{n,1}$ and $\dot{O}_{n,3} \leq \ddot{O}_{n,3}$. Hence, by (9) and (10),

$$O_n = O_{n,1} + O_{n,3} \leq 3\,(\ddot{O}_{n,1} + \ddot{O}_{n,3}).$$

Therefore by (9), $O_{n,1} \geq \ddot{O}_{n,1} + \ddot{O}_{n,3} \geq O_n/3$. Similarly, $O_{n,3} \geq O_n/3$. Using the fact that $O_n = O_{n,1} + O_{n,3}$, we actually get

$$\frac{O_n}{3} \leq O_{n,1} \quad O_{n,3} \leq \frac{2O_n}{3}.$$

This and (8) imply that

$$\frac{O_{n-1}}{3} \le \dot{O}_n \quad \dot{E}_n \le \frac{2O_{n-1}}{3}. \tag{11}$$

We are now ready to prove the result that leads to improved bounds for T_n.

Theorem 2. *Let*

$$\beta = (13 - \sqrt{145})/2$$

be one solution to the quadratic equation $(6+x)/(14-x) = x$. *Then,*

$$\beta\, T_n \le O_n \le (1-\beta)\, T_n. \tag{12}$$

Proof: We use induction on n. The case $n = 1$ is clear. Using (2), (4), (11) and the induction hypothesis, we have

$$E_n = 2\,(\ddot{E}_n + \ddot{O}_n) + \dot{E}_n \le 2\,T_{n-1} + 2O_{n-1}/3$$

$$\le (2 + 2(1-\beta)/3)T_{n-1}.$$

From (3), (4), (11) and the above inequality,

$$O_n = 2\,(\ddot{E}_n + \ddot{O}_n) + \dot{O}_n \ge 2\,T_{n-1} + O_{n-1}/3 \ge$$

$$(2+\beta/3)T_{n-1} \ge \frac{2+\beta/3}{2+2(1-\beta)/3}E_n = \frac{6+\beta}{8-2\beta}(T_n - O_n).$$

Simplifying we get

$$O_n \ge \frac{6+\beta}{14-\beta}T_n = \beta T_n.$$

Computing the lower bound for E_n in exactly the same manner we also get

$$E_n \ge \beta T_n.$$

Using $O_n = T_n - E_n$, we get the desired upper bound for O_n. This proves the result. □

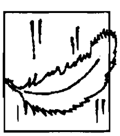
From (1) and (12), we get

$$(4+\beta)\,T_n \leq T_{n+1} \leq (5-\beta)\,T_n.$$

Since $T_1 = 4$, we infer by induction that for all n,

$$4(4.479)^{n-1} \leq T_n \leq (4.521)^n \qquad (13)$$

where the values are given to 3 decimals. Note that the lower and upper bounds are centred (logarithmically) around 4.5^n. We conjecture that $T_n = O(4.5^n)$. Assuming that the multiples of 2^n are uniformly distributed in the set $\mathcal{S}_n$, we would expect $T_n \approx |\mathcal{S}_n|/2^n = 4.5^n$ which explains the rationale behind our conjecture.



Seeing is not believing!

Consider a square whose sides are 4 units long. In the four corners, put in circles of unit radius. Each of these touches two others and there is an empty space at the centre of the square. With the centre of the square as its centre, draw a small circle in this empty space and gradually expand it until it touches one of the circles at the corner. By symmetry, it will touch all four of them. The radius of the inner circle is easily computed and found to be $\sqrt{2}-1$. Let us do a similar thing with the cube in three dimensions with each side of 4 units. Now there are 8 spheres at the 8 corners and the innermost sphere is found to have radius $\sqrt{3}-1$. Proceeding in this manner, let us consider the cuboid in 10 dimensions, say. Then, the innermost hypersphere has radius $\sqrt{10}-1$ which is bigger than 2. In other words, this innermost sphere protrudes out of the cuboid in some places! This is no paradox but merely a pointer to the fact that in higher dimensions, our intuition could mislead us.

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