A Geometry in which all Triangles are Isosceles

An Introduction to Non-Archimedean Analysis

P N Natarajan and K N Ranganathan

P N Natarajan is currently a Reader at the Ramakrishna Mission Vivekananda College, Chennai. His research interests are analysis, functional anlysis – both classical and p-adic.

K N Ranganathan is in the Department of Mathematics, Ramakrishna Mission, Vivekananda College, Chennai. He is interested in geometry, topology and problem solving.

The purpose of this article is to introduce 'a new analysis' to students of mathematics at the undergraduate and postgraduate levels, which in turn introduces a geometry very different from our Euclidean geometry and Riemannian geometry. Some strange things happen: for instance, 'every triangle becomes isosceles!'.

Analysis is that branch of mathematics where the idea of limits is used extensively. A study of analysis starts with limits, continuity, derivatives and goes on and on; almost all mathematical models are governed by differential equations over the field of real numbers. The real number line has a geometry which is Euclidean. Imagine a small pygmy tortoise trying to travel along a very long path; assume that its destination is at a very long distance from its starting point. If at each step the pygmy tortoise covers a small distance ε, can it ever reach its destination? (Assume that the tortoise has infinite life!): our common sense says 'yes'; 'yes', it is one of the important axioms in Euclidean geometry that "*Any large segment on a straight line can be covered by successive addition of small segments along the same line*" Equivalently it says that "*given any number $M > 0$, there exists an integer N such that $N > M$*" This is familiarly known as the Archimedean axiom of the real number field. What happens if we do not have this axiom? Can we have fields that are non-archimedean? We will see in this article that such fields do exist. However the metric on such fields introduces a geometry very different from our familiar Euclidean geometry and Riemannian geometry. Every triangle becomes isosceles! For any two open spheres, either one is contained in the other or the two spheres are disjoint.

We will be studying more of such geometry and the subsequent changes in limits of sequences and convergence of series. Finally we also observe that some natural phenomena are better explained and justified using analysis on such fields.

Now let us make a revisit to the field **Q** of rational numbers. We have

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

and we know how to get real numbers from rational numbers. We recall that every real number has a decimal expansion and that any real number can be written in the form

$$x = \pm 10^{\beta}(x_0 + x_1 10^{-1} + x_2 10^{-2} + \quad .), \beta \in Z$$

$$\text{and } x_j = 0, 1, \quad , 9, x_0 \neq 0 \quad (*)$$

Further, we note that the decimal expansion of a real number x enables us to look at the number x as the limit of a sequence of rational numbers. The metric space $(\mathbf{Q}, |\ |)$, where $|\ |$ is the usual metric on **Q**, is not complete in the sense that there are Cauchy sequences in **Q** which are not convergent in **Q**. We take the set ζ of all such Cauchy sequences in **Q** which do not have a limit in **Q** and consider two sequences $\{x_n\}$ and $\{y_n\}$ in ζ as equivalent if the sequence $\{x_n - y_n\}$ converges to zero in $(\mathbf{Q}, |\ |)$. After this identification, the set of equivalence classes in ζ is precisely the set **R** of real numbers. Now, the set **Q** of rational numbers is dense in **R** and the metric in **R** is given as follows: distance between x and $y = \lim |x_n - y_n|$ where $\{x_n\}$ and $\{y_n\}$ are the Cauchy sequences of rational numbers representing the real numbers x and y. This process of embedding

$\mathbf{Q}$, as a dense subspace, in a complete metric space $\mathbf{R}$ is called completion of the metric space $\mathbf{Q}$. In fact, we can imitate this process and embed any given metric space in a complete metric space and speak of the completion of a metric space.

By a valuation of a field $\mathbf{k}$, we mean a mapping $|\ |: \mathbf{k} \to \mathbf{R}$ (the field of real numbers) such that

$$|x| \geq 0; |x| = 0 \Leftrightarrow x = 0 \tag{1}$$

$$|xy| = |x||y| \tag{2}$$

$$|x + y| \leq |x| + |y| \text{ triangle inequality} \tag{3}$$

$x, y \in \mathbf{k}$. The field $\mathbf{k}$, with a valuation $|\ |$, is called a valued field.

We now give a few examples:

1. The usual absolute value function is a valuation of $\mathbf{Q}$ (the field of rational numbers);

2. Given any field $\mathbf{k}$, we can define a valuation of $\mathbf{k}$ as follows:

$$|x| = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

If $\mathbf{k}$ is a valued field with valuation $|\ \ |$, define

$$d(x, y) = |x - y| \tag{4}$$

$x, y \in \mathbf{k}$. One can easily show that d is a metric on $\mathbf{k}$ so that $\mathbf{k}$ is a metric space with respect to the 'metric induced by the valuation'defined by (4). As is done for general metric spaces, we can introduce topological concepts like open set, closed set, convergence, etc. in valued fields.

We start with $\mathbf{Q}$ and proceed in a different direction. Let c be a fixed real number with $0 < c < 1$ and let p

be a fixed prime number. We define a valuation $| \ |_p$ of $\mathbf{Q}$ thus:

If $x \in \mathbf{Q}, x \neq 0$, we write x in the form

$$x = p^{\alpha}\left(\frac{a}{b}\right)$$

where α, a, b are integers, p does not divide α and p does not divide b. We now define

$$|x|_p = c^{\alpha}$$

We also define $|0|_p = 0$.

It is now worthwhile checking that $| \ |_p$ is a valuation of $\mathbf{Q}$. It is clear, from the definition, that

$$|x|_p \geq 0 \text{ and } |x|_p = 0 \Leftrightarrow x = 0.$$

If $y = p^{\beta}(a'/b')$ is a rational number where β, a', b' are integers, p does not divide a' or b', we have

$$xy = p^{\alpha+\beta}\left(\frac{aa'}{bb'}\right)$$

where p does not divide aa' or bb', this being so since p is a prime number (In this context, it is worth examining why p was chosen to be a prime number!). Thus

$$|xy|_p = c^{\alpha+\beta} = c^{\alpha}c^{\beta} = |x|_p|y|_p.$$

We now prove that

$$|x+y|_p \leq \max(|x|_p, |y|_p) \tag{5}$$

a much stronger statement than (3). To prove (5), we show that

$$|x|_p \leq 1 \Rightarrow |1+x|_p \leq 1. \tag{6}$$

Leaving out the trivial case, let $x \neq 0$. From the definition of $| \, |_p$, it follows that $|x|_p \leq 1 \Rightarrow \alpha \geq 0$ and x can be written as

$$x = c'/d',$$

where c', d' are integers which are relatively prime and p does not divide d'. Then

$$1 + x = 1 + \frac{c'}{d'} = \frac{c' + d'}{d'}.$$

Noting that $1 + x$ has a denominator prime to p, we have,

$$|1 + x|_p \leq 1$$

We now prove (5). If $y = 0$, (5) is trivially true. Let us suppose that $y \neq 0$. Without loss of generality, we shall suppose that $|x|_p \leq |y|_p$ so that $|x/y|_p \leq 1$.

In view of (6), we have,

$$|1 + x/y|_p \leq 1, \text{i.e, } |x + y|_p \leq |y|_p = \max(|x|_p, |y|_p)$$

which completes the proof of the claim.

If a valuation of **k** satisfies (5) too, it is called a 'non-archimedean valuation' of **k** and **k**, with such a valuation, is called a 'non-archimedean valued field' or just a 'non-archimedean field'. The metric induced by a non-archimedean valuation satisfies the much stronger inequality

$$d(x, y) \leq \max(d(x, z), d(z, y)), \tag{7}$$

which is known as the 'ultrametric inequality' (In fact any metric which satisfies (7) is called an 'ultrametric'). We have thus proved that $| \, |_p$ is a non-archimedean valuation of the rational number field $\mathbf{Q}$. The valuation $| \, |_p$ is called the p-adic valuation of $\mathbf{Q}$. The completion of $\mathbf{Q}$ with respect to the p-adic valuation is called the p-adic field, denoted by $\mathbf{Q}_p$.

The completion $\mathbf{Q}_p$ of $(\mathbf{Q}, |\ |_p)$ resembles the completion $\mathbf{R}$ of $(\mathbf{Q}, |\ |)$ in the sense that every element $x \neq 0$ in the p-adic field $\mathbf{Q}_p$ has a unique infinite expansion

$$x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + x_3 p^3 + \quad .)$$

where $\gamma = \gamma(x) \in \mathbf{Z}$ and x_j are integers such that $0 \leq x_j \leq p-1, x_0 > 0$, very similar to the decimal expansion of any real number as in (*). Further,

$$|x|_p = c^{\gamma}.$$

We will come back to this analogy a little later.

We are familiar with analysis in $\mathbf{R}$ (real analysis) or in $\mathbf{C}$ (complex analysis). In the present article, we prove some elementary results in analysis in non-archimedean fields, referred to as non-archimedean analysis or p-adic analysis. These results point out significant departures from real or complex analysis. In the sequel we shall suppose that $\mathbf{k}$ is a non-archimedean field.

Theorem: If $\mathbf{k}$ is a non-archimedean field and if $|a| \neq |b|, a, b \in \mathbf{k}$, then

$$|a+b| = \max(|a|, |b|).$$

Proof. For definiteness, let $|a| > |b|$. Then

$$|a| = |(a+b) - b| \leq \max(|a+b|, |b|) = |a+b|$$

for, otherwise, $|a| \leq |b|$, which is a contradiction of our assumption. Thus,

$$|a| \leq |a+b| \leq \max(|a|, |b|) = |a|.$$

Consequently

$$|a+b| = |a| = \max(|a|, |b|).$$

Corollary: Any triangle is **isosceles.**

Proof. Let d be the ultrametric induced by the non-archimedean valuation $|\ |$ in $\mathbf{k}$. Consider a triangle with vertices x, y and z. If the triangle is equilateral i.e., $d(x,y) = d(y,z) = d(z,x)$, then there is nothing to prove! If the triangle is not equilateral (say) $d(x,y) \neq d(y,z)$, then we have

$$d(x,z) = |x-z| = |(x-y)+(y-z)| =$$

$$\max(|x-y|, |y-z|) = \max(d(x,y), d(y,z)),$$

which establishes our claim. In this context we shall give one more interesting result.

Theorem: Every point of the open sphere

$$\mathbf{S}_\varepsilon(x) = \{y \in k | \ \|y-x\| < \varepsilon\}$$

is a centre i.e., if $y \in \mathbf{S}_\varepsilon(x), \mathbf{S}_\varepsilon(x) = \mathbf{S}_\varepsilon(y)$.

Proof. Let $y \in \mathbf{S}_\varepsilon(x)$ and $z \in \mathbf{S}_\varepsilon(y)$. Then $|y-x| < \varepsilon$ and $|z-y| < \varepsilon$ so that

$$|z-x| = |(z-y)+(y-x)| \leq \max(|z-y|, |y-x|) < \varepsilon.$$

Thus $z \in \mathbf{S}_\varepsilon(x)$. Consequently $\mathbf{S}_\varepsilon(y) \subset \mathbf{S}_\varepsilon(x)$. The reverse inclusion can be similarly proved so that $S_\varepsilon(x) = S_\varepsilon(y)$, completing the proof.

Corollary: Any two spheres are either disjoint or identical.

Theorem: A sequence $\{x_n\}$ in a non-archimedean field $\mathbf{k}$ is a Cauchy sequence if and only if

$$|x_{n+1} - x_n| \to 0, n \to \infty. \tag{8}$$

Proof. If $\{x_n\}$ is a Cauchy sequence in $\mathbf{k}$, it is clear that (8) holds. If (8) holds, given $\varepsilon > 0$, there exists a positive integer N such that

$$|x_{n+1} - x_n| < \varepsilon, n \geq N. \tag{9}$$

If $m > n$,

$$|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)| \leq \max(|x_m - x_{m-1}|, |x_{m-1} - x_{m-2}|, \ldots, |x_{n+1} - x_n|) < \varepsilon$$

in view of (9), which proves that $\{x_n\}$ is a Cauchy sequence.

In real or complex analysis, we know that if Σx_n converges, then $x_n \to 0, n \to \infty$. However the converse is not true. For instance $\Sigma(1/n)$ diverges, though $1/n \to 0, n \to \infty$. We will now show that the converse is also valid in the case of complete, non-archimedean fields.

Theorem: Let $\mathbf{k}$ be a complete field with respect to a non-archimedean valuation $|\ \ |$. Then $\Sigma a_n, a_n \in \mathbf{k}, n = 1, 2, \ldots$ converges if and only if $a_n \to 0, n \to \infty$.

Proof. Let Σa_n converge and $s_n = a_1 + a_2 + \ldots + a_n, n = 1, 2, \ldots$. Then $\{s_n\}$ converges and so

$$|a_n| = |s_n - s_{n-1}| \to 0, n \to \infty$$

so that $a_n \to 0, n \to \infty$. More importantly, let us prove the converse. Assume that $a_n \to 0, n \to \infty$. Now,

$$|s_n - s_{n-1}| = |a_n| \to 0, n \to \infty$$

so that $\{s_n\}$ is a Cauchy sequence in view of Theorem 3. Since $\mathbf{k}$ is complete, $\lim s_n$ exists in $\mathbf{k}$ and so Σa_n converges in $\mathbf{k}$.

Let us have another look at the p-adic field $\mathbf{Q_p}$. As we have already seen, any p-adic number $x \neq 0$ can be uniquely written as

$$x = p^{\gamma}(x_0 + x_1 p + x_2 p^2 + x_3 p^3 + \ldots)$$

where $\gamma = \gamma(x) \in \mathbf{Z}$ and x_j are integers such that $0 \leq x_j \leq p - 1, x_0 > 0$. Such a series is always convergent in $\mathbf{Q_p}$; for, by Theorem 4, the series is convergent if and only if the sequence $\{p^\gamma x_n p_n^n\}$ tends to zero as n tends to infinity and this is in fact so because $|p^\gamma x_n p^n| = c^{\gamma+n}$; since $0 < c < 1, c^{\gamma+n}$ tends to zero as n tends to infinity. If $\gamma = \gamma(x) \geq 0, x$ is a p-adic integer. The p-adic integers are characterized by $|x|_p \leq 1$. Actually any p-adic number can be written in the form

$$x = p^\gamma(x_0 + x_1 p + x_2 p^2 + x_3 p^3 + \quad .)$$

where $\gamma \in \mathbf{Z}$ and $x_j \in \mathbf{Z}$ and $|x|_p = c^\gamma$. The integer coefficients are unique modulo p. For example, the canonical p-adic expansion of -1 is given by

$$-1 = p - 1 + (p-1)p + (p-1)p^2 + (p-1)p^3 +$$

It is clear that the rational numbers which are p-adic integers are those which can be written in the form m/n with g.c.d$(n,p) = 1$. In $\mathbf{Q}_5$, the rational number $1/5$ is written as

$$1/5 = 1(5^{-1} + 0 + 0(5^1) + 0(5^2) +$$

The p-adic number

$$x = a_{-n}p^{-n} + a_{-n+1}p^{-n+1} + \qquad + a_0 + a_1 p + a_2 p^2 +$$

is written as

$$x = a_{-n}a_{-n+1}a_{-n+2} \qquad a_0, a_1 a_2 a_3 \qquad (p).$$

In this notation, we have

$$1/5 = 10,000. \quad (5).$$

Again in $\mathbf{Q}_5$, the rational number $3/8$ is a 5-adic integer since g.c.d.(8,5) $= 1$. The 5-adic expansion of $3/8$ is got as follows. First we observe that

$$|3/8|_5 = |5|_5^0 = 1.$$

So,

$$3/8 = a_0 + a_1 5 + a_2 5^2 +$$

Solving $8x = 1 \bmod(5)$, we get $x = 2$. Therefore, $3/8 - a_0 = 0 \bmod(5)$ implies that $a_0 = 2.3 \bmod(5) = 1 \bmod(5)$. Now, $3/8 - 1 = -5/8$ and again

$$|-5/8|_5 = |5|_5^0 = 1$$

gives $a_1 = (-5/8)(1/5) = (-1/8) \bmod(5)$. We have 2.8 = 1 mod(5) and therefore $a_1 = 2(-1) = 3 \bmod(5)$. Now, $3/8 - 1 - 3.5 = -5/8 - 15 = -125/8$ and

$$|-125/8|_5 = |5|_5^3.$$

This means that $a_2 = 0$ and $a_3 \neq 0$. Since $(3/8 - 1 - 3.5)/5^3 = (-125/8)/5^3 = -1/8$, we get $a_3 = 3$ as before. Thus proceeding we get

$$3/8 = a_0 + a_1 5 + a_2 5^2 + \quad = 1,30303030\,. \quad (5)$$

The above example explains how to get p-adic expansions of rational numbers. It is not difficult to prove that

$$x = a_{-n} a_{-n+2} a_{-n+2} \quad a_0, a_1 a_2 a_3 \quad (p)$$

is rational if and only if the sequence $a_1 a_2 a_3$ is ultimately periodic. We might have defined $\mathbf{Q_p}$ as the set of all formal power series $\{\sum_{k=n}^{\infty} a_k p^k | a_k \in Z\}$ *where two series* $\sum a_k p^k$ *and* $\sum b_k p^k$ *are equal iff their partial sums* S_j *= sum of the first* j *terms of* $\sum a_k p^k$ *and* T_j *= sum of the first* j *terms of* $\sum b_k p^k$ *satisfy* $S_j - T_j = \frac{u}{v}\, p^j$ *for all sufficiently large* j *and* $\frac{u}{v} \in Q$ *such that* $g.c.d.(p, v) = 1$

Also, we could have defined addition and multiplication in the usual natural fashion. These discussions suggest how solutions of a congruence modulo arbitrarily large powers of a prime p lead to p-adic numbers.

Going deeper into the topic of p-adic analysis, one can point out more and more results, which indicate significant departures from real and complex analysis – in fact p-adic analysis is a strange terrain. One may ask, 'Are there natural non-archimedean phenomena?' We have observed as a corollary of Theorem 2 that any two spheres in a non-archimedean field k are either disjoint or identical. This situation is amply illustrated by two drops of mercury on a surface, which is a typical non-archimedean phenomenon.

In conclusion we point out that the general directions in which p-adic analysis is applied in mathematical physics are: geometry of space-time at small distances; classical and quantum chaos-investigation of complicated systems such as spin glasses and fractals; stochastic processes and probability theory; connections with q-analysis and quantum groups; extensions of the formalism of theoretical physics to other number fields; adelic formulas giving a decomposition of certain physical systems into more simple parts etc. Further, p-adic analysis is likely to have applications to the theory of turbulence, biology, dynamical systems, computers, cryptography, problems of information transference and other natural sciences that study systems with chaotic fractal behaviour and hierarchical structures. For a more detailed and in-depth study of p-adic analysis and its applications, one can refer to [1],[2] and [3].

Suggested Reading

[1] G Bachman, *Introduction to p-adic numbers and valuation theory*, Academic Press, 1964.

[2] L Narici, E Beckenstein and G Bachman, *Functional analysis and valuation theory*, Marcel Dekker, 1971.

[3] V S Vladimirov, I V Volovich and E I Zelenov, *p-Adic analysis and mathematical physics*, World Scientific, 1994.

Address for Correspondence
P N Natarajan and
K N Ranganathan
Department of Mathematics
Ramakrishna Mission
Vivekananda College
Chennai 600 004, India.