Classroom

In this section of* Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Ampére versus Biot–Savart

The International Physics Olympiad was held in Padua, Italy during July 18-27 1999. It marked India's second foray into this exciting event where sixty-two nations participated. As the pedagogical leader of the Indian team at Padua, I was privileged to be in the thick of action. Our performance was creditable: all our participants won medals, garnering four silvers and one bronze. Out of the five special prizes, one went to a member of our team. This article describes an interesting and historically rich problem that was posed as part of the five hour theoretical examination.

Preamble

The International Physics Olympiad is an annual event initiated by the erstwhile East European nations three decades ago. India has been a late-comer to this event which is part festival, part competition – in short a celebration of pre-college physics. We participated for the first time last year. This year was our second foray into this exciting event. Participating for the second time, our team performed creditably, winning four silver medals and one bronze medal. All our five students secured medals. In addition to the medals, we secured one of the five special prizes awarded by the scientific committee of the XXX Physics Olympiad. This special prize was for the best solution to a problem based on a famous debate on electromagnetism in the last century (problem no. 2 in the five hour long theoretical exam), and it was secured by one of our students: Raju Suvrat. This article describes this historically interesting problem and its solution.

Ampere versus Biot–Savart: The Problem

Among the first successes of the interpretation of magnetic phenomena by Ampère was the computation of the magnetic field B generated by wires carrying an electric current as contrasted with the earlier erroneous predictions of Biot and Savart. This problem is concerned with this famous debate. Ampère was vindicated and his work was later embodied in Maxwell's electromagnetic theory and is universally accepted.

Consider a very long thin wire, carrying a current i, bent into the form of a 'V', with an angular half-span α radians (see *Figure* 1).

Using our contemporary knowledge of electromagnetism,

1. Find the direction of the field B at point P on the

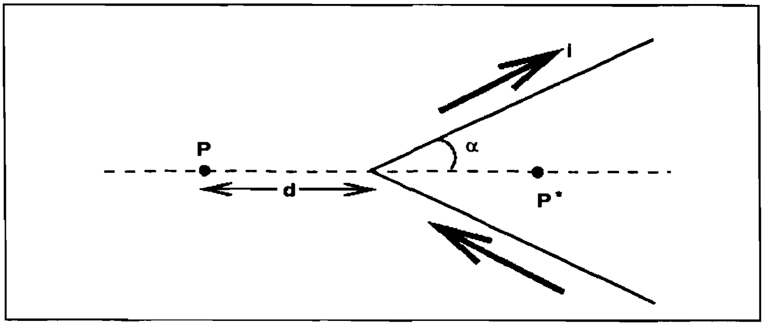

Figure 1.

axis of the 'V' shaped wire and outside of it. The distance between the vertex and the point P is d.

2. Accepting that the field is proportional to $\tan(\alpha/2)$ as suggested by Ampère, obtain the proportionality factor k in $|\vec{B}(\mathrm{P})| = k\tan(\alpha/2)$.

3. Compute the field B at the point P* symmetric with respect to the vertex, i.e. along the axis and at the distance d, but inside the 'V' (see *Figure* 1).

4. In order to measure the magnetic field, we place at P a small magnetic needle with moment of inertia I and magnetic dipole moment μ. It oscillates in a plane containing the direction of B. Its axis of oscillation is fixed at P. Compute the period of small oscillations of this needle.

5. Under the same conditions Biot and Savart had instead assumed that the magnitude of the magnetic field at P was (we use here the modern notation) $B(\mathrm{P}) = i\mu_o\alpha/\pi^2 d$, where μ_o is the magnetic permeability of vacuum. In fact they attempted to distinguish with an experiment between the two interpretations (that of Ampère and of Biot and Savart) by measuring the oscillation period of the magnetic needle as a function of the 'V' span. For some α values, however, the difference between the two predictions is too small to be easily measurable.

   In order to distinguish experimentally between the two predictions for the needle's oscillation period T at P, we need a difference of at least 10 % ($T_1 > T_2$ where T_1 is the Ampère prediction and T_2 is the Biot–Savart prediction). State the range of α that allows us to distinguish between the two interpretations.

Ampère versus Biot–Savart: The Solution

1. The contribution to B by each leg of the 'V' has the same direction as the corresponding infinite wire. If the current proceeds as indicated by the arrow, the magnetic field is orthogonal to the wire plane which we take here as the plane of the paper. B(P) is directed out of the plane of the paper and towards us.

2. We can solve this part by a number of ways. The simplest is to consider the limiting case $\alpha = \pi/2$. In this case the 'V' becomes a straight infinite wire. The magnitude of the field is

$$B(\mathrm{P}) = \frac{i\mu_0}{2\pi d}$$

since $\tan(\pi/4) = 1$, the factor k is

$$k = \frac{i\mu_0}{2\pi d}. \qquad (1)$$

An alternative but perhaps painful way is to recall the formula for the magnetic field of a finite stretch of wire at a point P which is at a perpendicular distance h from the wire,

$$\mathrm{B} = \frac{i\mu_0(\cos\theta_1 - \cos\theta_2)}{4\pi h} \qquad (2)$$

where θ_1 and θ_2 are the angles made by the ends of the finite wire with the point P. Each leg of the wire produces a field

$$B_1 = \frac{i\mu_0(1 - \cos\alpha)}{4\pi d \sin\alpha} = \frac{i\mu_0 \tan(\alpha/2)}{4\pi d}.$$

By symmetry, the total field is twice that generated by each leg and thus we once again obtain (1).

3. Once again there are a number of ways by which one may obtain the solution to this part. The solution which employs symmetry considerations is the most elegant. To compute $B(\mathrm{P}^*)$ we consider the 'V' as equivalent to two crossed wires (a and b) carrying a current i (thick arrows in the figure) plus another 'V' shaped wire denoted as 'V′' and carrying the same current i but in the opposite direction (thin arrows in the figure).

   Thus $B(\mathrm{P}^*) = B_a(\mathrm{P}^*) + B_b(\mathrm{P}^*) + B_{V'(\mathrm{P}^*)}$. The individual contributions are:

$$B_a(\mathrm{P}^*) = B_b(\mathrm{P}^*) = \frac{i\mu_0}{2\pi d \sin\alpha}$$

into the plane of the paper.

$$B_{V'(\mathrm{P}^*)} = \frac{i\mu_0 \tan(\alpha/2)}{2\pi d}$$

out of the plane of the paper.

This yields,

$$B(\mathrm{P}^*) = \frac{i\mu_0}{2\pi d}\left[\frac{2}{\sin\alpha} - \tan(\frac{\alpha}{2})\right] = k\cot(\frac{\alpha}{2})$$

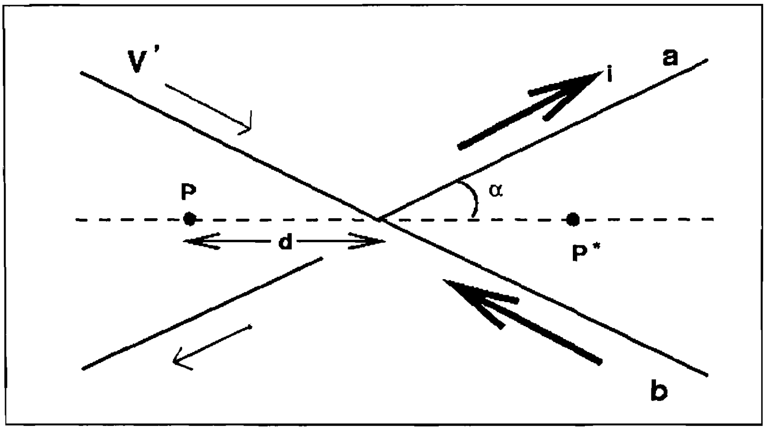

Figure 2.

and the field is perpendicular to and into the plane of the paper.

One may also obtain the solution by the following interesting exercise. The point P* inside a 'V' with a half span α can be treated as if it would be on the outside of a 'V' with half-span $\pi - \alpha$ carrying the same current but in the opposite way, therefore the field is

$$\begin{aligned} B(\mathrm{P}^*) &= k\tan(\frac{\pi-\alpha}{2}) \\ &= k\cot(\frac{\alpha}{2}) \end{aligned}$$

This field is perpendicular to the plane of the paper but into it since the current is flowing in the opposite direction.

Of course one can obtain the same answer by the somewhat laborious process of applying the formula for the magnetic field of a finite current carrying wire (2).

4. The torque acting on the magnetic needle placed at P is $\vec{\tau} = \vec{\mu} \times \vec{B}$. If the needle is displaced from its equilibrium position by a small angle β so that we may approximate $\sin(\beta) \approx \beta$ we have from Newton's second law,

$$\begin{aligned} \tau &= -\mu B\beta \\ &= I\frac{d^2\beta}{dt^2}. \end{aligned}$$

The period T of small oscillations is then

$$T = 2\pi\sqrt{\frac{I}{\mu B}}. \tag{3}$$

5. The Ampère and Biot–Savart predictions are respectively

$$B_A(\mathrm{P}) = \frac{i\mu_0\tan(\alpha/2)}{2\pi d} \tag{4}$$

$$B_{BS}(\mathrm{P}) = i\mu_o\alpha/\pi^2 d. \tag{5}$$

Using (3) the respective time periods are

$$T_A = 2\pi\sqrt{\frac{2\pi Id}{\mu_o\mu i\tan(\frac{\alpha}{2})}} \quad (6)$$

$$T_{BS} = 2\pi\sqrt{\frac{\pi^2 Id}{\mu_o\mu i\alpha\tan\alpha}}. \quad (7)$$

For $\alpha = \pi/2$ (maximum possible value), $T_A = T_{BS}$; and for $\alpha \longrightarrow 0, T_A = 2T_{BS}/\sqrt{\pi} \approx 1.128\, T_{BS}$. Note that in this range $\tan(\alpha/2)/(\alpha/2)$ is a monotonic function, such that T_A/T_{BS} is a monotonically decreasing function of α. Given the experimental conditions in the nineteenth century it was not possible to distinguish between the two interpretations if the value of α was larger than the value for which $T_A < 1.1\ T_{BS}$, i.e less than 10 % difference. This happens at,

$$\tan(\frac{\alpha}{2}) = \frac{4}{1.21\pi}\frac{\alpha}{2}. \quad (8)$$

By looking at the trigonometry tables or by using a calculator we obtain that α must be less than 0.77 rad or 44°. A series solution where we keep the first two terms,

$$\tan(\frac{\alpha}{2}) \approx \frac{\alpha}{2} + \frac{1}{3}(\frac{\alpha}{2})^2$$

yields the erroneous answer $\alpha \approx 62.7^o$. A graphical solution is also feasible.

This interesting problem is a slightly modified version of the one which appeared in the XXX International Physics Olympiad held in Padua, Italy, July 1999. As mentioned earlier the award for the best solution to this problem went to Raju Suvrat, a participating student member of the Indian team. He employed the method of Regula Falsi to solve the nonlinear equation (8).

We have a related question to pose for the reader. The Ampère expression for the magnetic field (4) can be obtained from the basic expression for the magnetic field due to a current carrying element $\vec{dl}$,

$$d\vec{B} = \frac{\mu_o i}{4\pi}\vec{dl} \times \frac{\vec{r}}{r^3}$$

which in modern textbooks is called the Biot–Savart law. What is the corresponding microscopic law which would yield (5)? We invite the reader to carry out this 'inverse' exercise. [Note: Guessing or proving the underlying microscopic law from a phenomenological law or laws is normally a difficult exercise and is known in the literature as the 'inverse problem'.]

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Suggested Reading

[1] H C Pradhan, The 29th International Physics Olympiad, *Physics News*, Vol. 29, 151, 1998.

[2] H C Pradhan, Report on the First National Physics Olympiad and the 29th International Physics Olympiad, *Resonance*, Vol. 4,No.2 , p.104, 1999.

[3] *Proceedings XXIX International Physics Olympiad*, edited by V Augustsson (XXIX International Physics Olympiad Committee, Taeknigardor, Dunhagi, 5 Iceland, 1998).