Bernoulli's Equation

Jaywant H Arakeri

Jaywant Arakeri is in the
Mechanical Engineering
Department and in the
Centre for Product
Development and
Manufacture at the Indian
Institute of Science,
Bangalore. His research is
mainly in instability and
turbulence in fluid motion.
He is also interested in
issues related to environ-
ment and development.

Introduction

Fluid dynamics is concerned with the flow of liquids and gases. Fluid flow is everywhere around us: in rivers, in the atmosphere, within our bodies, in plants, in engines and pumps, around moving bodies like fish, birds, ships, airplanes and cars. Thus fluid mechanics is an important area of study for engineers and scientists. Bernoulli's equation is among the first things, or sometimes the only thing, a person learns about fluid mechanics. In this article we look at this equation, first given in statement form by Daniel Bernoulli in 1738, and in a mathematical form by Leonhard Euler only later in 1755.

A fluid is composed of molecules moving randomly. To fix ideas let us consider gases. (Many of the conclusions regarding nature of forces hold equally for liquids, even though the mechanisms at a molecular level are different in gases and liquids.) A very small volume of gas contains a large number of molecules. For example, at ordinary conditions of temperature and pressure, air in a box one micron in length contains about 10^8 molecules. The molecules move randomly and rapidly – the average molecular speed is about 300 m/s (about 1000 kmph). Thus a gas can be considered to be a large number of rigid balls moving and colliding against each other. The random motion and collisions transfer momentum, and cause pressure and viscous stresses (friction) in fluids.

One way to solve fluid mechanics problems is to start at the molecular dynamics level. However, the majority of flows we are interested in are at scales of a few millimeters (blood flow in an artery) to maybe a few kilometers (wind systems in the atmosphere) – much larger than the intermolecular spacing. Thus it makes sense to

solve the problem at a larger scale than the molecular spacing, but take into account the molecular motion in some way.

Here we come to an important concept of a fluid particle. A fluid particle is a very small, imaginary blob of fluid. It is big enough that it contains a very large number of molecules but still much smaller than the scales over which changes (for example, in velocity) occur; 0.1 mm is a good size to imagine. The velocity at a point in the fluid will be the velocity of the particle which will be the average velocity of all the molecules in the particle. Of course, fluid is one continuous medium; only we imagine the fluid to contain many many contiguous fluid particles to aid our understanding. We can also imagine an isolated fluid particle surrounded by the rest of the fluid (*Figure* A). Looking at the macroscopic picture and treating the fluid as continuous instead of discontinuous, which it is at a molecular level, is called the continuum hypothesis. It must be remembered that the molecular motion is not neglected – infact the exchange of molecules between a fluid particle and its surrounding fluid contributes to viscous stresses and heat conduction. In this article most of the discussion is in terms of the motion of fluid particles.

Motion in a fluid is usually set up by motion of some solid wall – rotating fan blades set up a breeze or walking sets the air around you into motion. The motion of the wall constrains the fluid next to the wall to move in a certain way. Fluid away from the wall in turn is set into
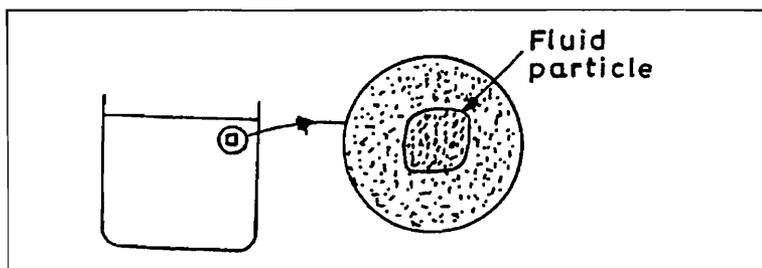

Figure A. A fluid particle is a very small imaginary blob of fluid, here shown schematically in a glass of water. The dots indicate molecules which move in and out of the particle.

motion through the action of pressure and viscosity. The nature and extent of the motion depends on the details of the motion. For example, from experience we know that a thin flat plate (a knife) in water moving in its own plane causes very little disturbance compared to when the plate is moved normal to its plane. A pertinent question is what causes the fluid to move. How does a fluid particle get information that it should move? Like rigid body motion fluid particle motion is governed by Newton's second law, $F=ma$. Therefore there must be some force on the fluid particle to make it move. We will see later what these forces are.

Motion of Disks

First we try to understand the nature of fluid motion with two examples. Consider a row of five (the precise number is not important) initially stationary carrom coins (disks) each with mass $M = 50$ gm, touching each other, lying on a horizontal surface and constrained to move in a slot (*Figure* B). Suppose we apply an inline force of say $F = 2.5$ Newtons at one end to the row of disks. The row will move with an acceleration $= F/5M = 10$ m/s^2 Each disk in the row will have this same acceleration. Thus the force on each disk will be $F/5 = 0.5$ N; F is equally divided among the five disks.

The disk closest to the applied force will have on one side the 2.5 N force, and on the other side a force from the second disc $= 2.0$ N and in the opposite direction. The second disk will have a force of 2.0 N on one side, will use 0.5 N of this force for its own acceleration and

Figure B. Motion of a line of five disks due to a force = 2.5 N applied at one end. Each disk has a net force = 0.5 N. For example, the second disk has a force of 2 N on one side and 1.5 N on the other side.

F = 2.5 N

2.0 N   1.5 N
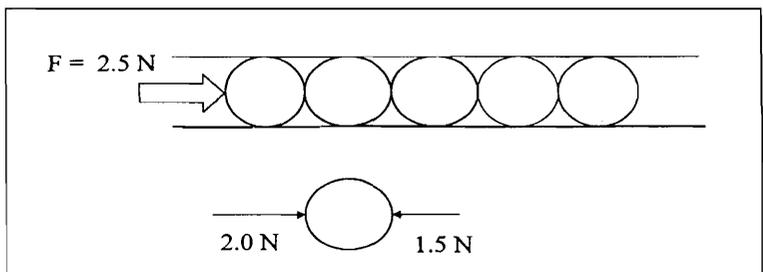
transmit 1.5 N (see *Figure* B) to the third disk, and so on. The important point is that since the disks are constrained to be in contact all the time, force and motion are transmitted from one disk to another.

Now consider a more complicated siuation. Instead of a parallel slot, we have two curved walls on a horizontal surface to form a narrowing passage. The passage is completely filled with disks (*Figure* C). The wider end can hold, say, nine disks along the edge and the narrower end can hold, say, three disks along the edge; width of the wider end is three times that of the narrower end. Let us say new disks are being pushed into the wider end at the rate of nine disks per second (Q_{in}). Since the passage is packed, some disks which were earlier in the passage will have to leave from the narrower end and the rate of leaving $Q_{out} = Q_{in}$, i.e., nine disks per second. (Q_{out} cannot be less than Q_{in} because then disks will accumulate in the passage which cannot happen as the passage is assumed to be packed; if $Q_{out} > Q_{in}$, then voids
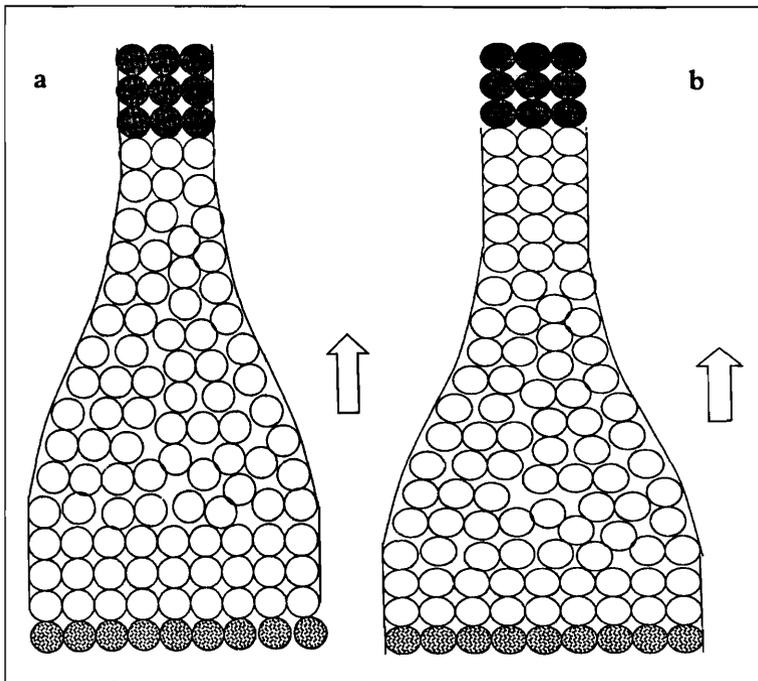

Figure C. Motion of disks within a passage. Disks are being fed at a steady rate on the left-hand side and leave on the right-hand side. Nine disks (dotted) on the left-hand side which were on the outside (a) enter the passage at a later time (b); during this time the same number of disks (shaded) leave on the right-hand side, but with three times the velocity.

voids will be created in the passage.) What about the velocity of the exiting disks in comparison with the velocity of the entering disks? Since the exit is three times narrower than the entrance, the velocity at the exit has to be three times the velocity at the entrance (*FigureC*).

As in the example with a single row of disks, here also motion and force at one point is transferred to disks at other points: pushing disks in at the wider end causes disks to leave at the narrower end and obviously causes motion of disks in between as well. The geometry of the walls not only determines the motion adjacent to the walls (that disks cannot enter a wall but atmost can slide along it) but influences motion away from the walls as well.

Continuous feeding of disks means that a disk which is at the entrance now will at a later time reach the exit. On the way its velocity (magnitude and direction) continously changes. These changes in velocity of the disk are due to forces from adjacent discs and, if in contact with one, from a wall.

If the disks are being fed in at a constant rate and we focus on some fixed point in the passage then the velocity of disks passing *through that point* will not change with time. When the velocity at any fixed location does not change with time it is called steady motion. However, if we focus our attention on a particular disk then of course as we have seen its velocity, in general, can change with time. Thus acceleration takes place even in steady motion, because the disk is moving through different points in space and the velocity changes with location. If the feed rate changes with time then the velocity at any point changes with time (and position) and we have unsteady motion.

Stream-tubes and Streamlines

Fluid flow is somewhat like the motion of the disks.

Fluid particles are like the disks, and like them transmit motion and forces to different points in the fluid. Pressure is mainly responsible for this transmission. (As we know pressure on a surface is normal force per unit surface area. In a non-viscous fluid, pressure at a point is the same in all directions.) It must be remembered that unlike disks, fluid particles can deform, and also there is exchange of molecules between contiguous fluid particles which is absent in the case of disks. Also as mentioned earlier we imagine the existence of fluid particles to aid our understanding; in reality it is, forgetting the molecules, one continuous medium.

Fluid flow is usually described in terms of the velocity (three components in general) with time at all points in space. The velocity at a point corresponds to the velocity of a fluid particle positioned at that point at that time. Velocity is with respect to some reference frame. Unlike in rigid body motion where the velocity of a particular body is given as a function of time, in fluid mechanics the velocity is given at fixed points as a function of time; any given point will see different fluid particles pass through it.

Like the passage through which the disks moved, in fluid flow we can imagine a fictitious tube through which the fluid moves and *across whose walls there is no flow*. Such a tube is called a stream-tube. We can imagine a fluid flow to be composed of a large number of streamtubes, touching each other. If we assume the fluid to be incompressible, i.e., density is a constant, then the volume flow rate (in litres per minute, for example) entering a tube = volume flow rate leaving the tube = Q. At any cross-section, $Q = $ (velocity) $\times$ (crossectional area of the tube); velocity $\sim 1/$crossectional area.

If we shrink a stream-tube to a line we get a streamline. A streamline gives the direction of velocity: the tangent to the streamline at any point is the direction of the

Figure D. Streamlines, in flow over a streamlined car model. Divergence of streamlines, as near the front of the car, indicates lower velocity and higher pressure. Convergence of streamlines near the top indicates higher velocity and lower pressure.
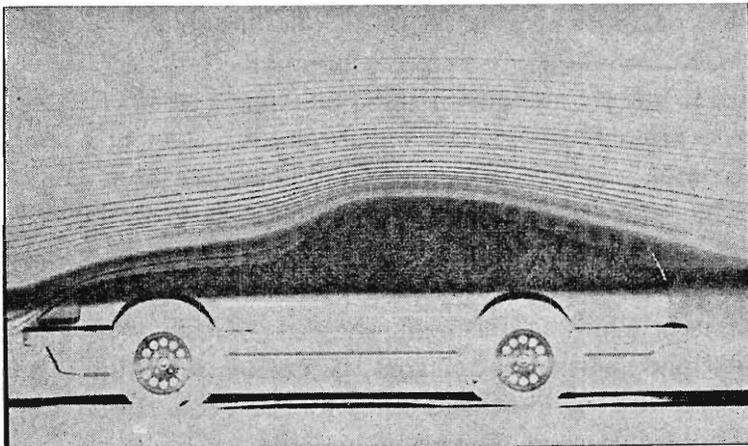
velocity at that point. Fluid flow is often depicted using streamlines. *Figure* D shows streamlines, in flow past a stationary car model. *Figure* E shows streamlines in flow past an airfoil. In the pictures the space between two adjacent streamlines is a stream-tube. A streamline picture gives important information about the flow field. Besides the direction of fluid velocity, the magnitude of the velocity can also be visualized.

Narrowing of a stream-tube, or equivalently streamlines bunching close together, indicates an increased fluid velocity as does the widening of a tube indicate reduced velocity. We will see later that from the streamline pic-

Figure E. Streamlines over an airfoil. Lift is produced by lower pressure (closely spaced streamlines) on the top and higher pressure (widely spaced streamlines) at the bottom.
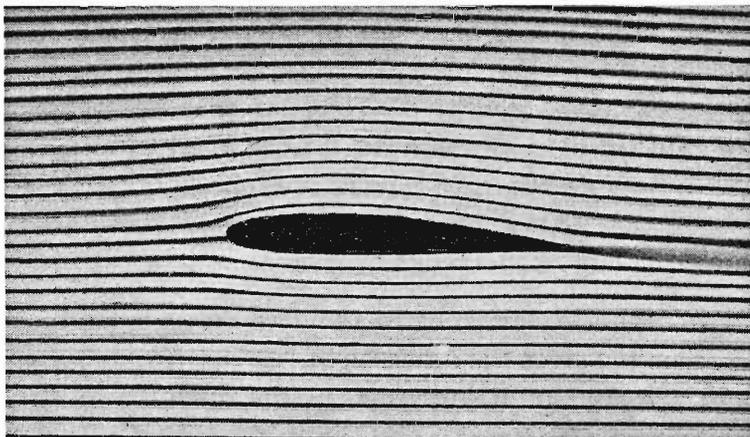
ture we can get an idea of the pressure variation in the flow.

Steady and Unsteady Flow

Steady flow is one in which the fluid velocity at any given point does not change with time. In unsteady flow, fluid velocity is a function of time. Flow around a body, say a bus, moving with constant speed in still air, gives an interesting example of steady and unsteady flow. To a passenger in the bus (reference frame fixed to the bus) the flow is nearly steady. If he stuck out his hand he would feel a nearly constant wind speed. Were he able to see, he would see fluid particles going past him (*Figure* F). To a roadside observer (reference frame fixed to the ambient), however, the flow is unsteady. When the bus is faraway there is no flow; as it comes near there is gust of air. The roadside observer would measure a velocity which changes with time. A fluid particle (which is stationary initially) is set into motion as the bus comes near, gets carried forward some distance, and comes to rest again as the bus moves away (*Figure* G). The fluid particle behaves like a piece of paper lying on the road. It must be made clear that the flow is exactly the same in the two cases. Only the references or points of view are different.

In steady flow the path of fluid particle will be along a streamline; and the streamline picture does not change with time. In unsteady flow particle paths in general are different from streamlines.
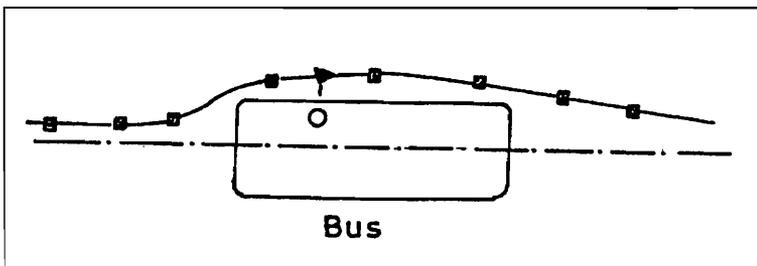

Figure F. A particle at different times as observed by a passenger in a bus. The line is the path taken by the fluid particle.
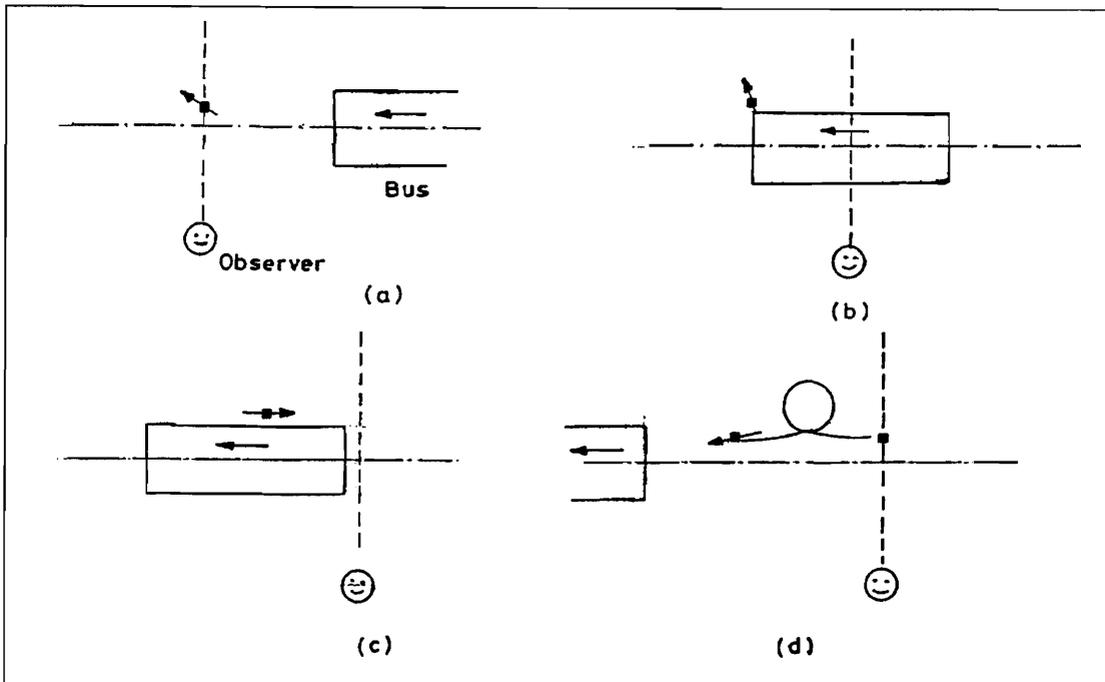
Figure G. The motion of a fluid particle as observed by a roadside observer as a bus goes past the observer. a) The initially stationary particle moves to the side and front as the bus approaches. The arrow indicates only direction of motion, and not the magnitude of velocity. b)The fluid particle is moving laterally. Note the particle has moved relative to its original position, which was on the dotted line. c) Particle motion is in opposite direction to that of the bus. d) The particle becomes stationary again. The loop indicates the path taken by the particle.

Bernoulli's Equation

Bernoulli's equation or theorem can be derived by either applying Newton's second law to a fluid particle or by using the equation of energy (First law of thermodynamics). Bernoulli's equation is derived assuming ideal flow, i.e., viscous forces are neglected. To obtain Bernoulli's equation we need expressions for acceleration and forces on a fluid particle. These are derived next.

Figure H. S is the coordinate along the streamline. Velocity U of a fluid particle is tangential to the streamline at that point.
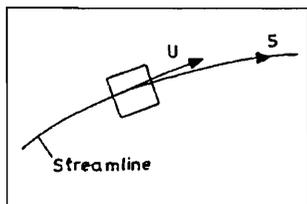
Acceleration of a Fluid Particle

Consider a fluid particle moving with velocity U along a streamline. Distance measured along the streamline is S (*Figure* H). A fluid particle which is at some location can accelerate due to two reasons. One is if fluid velocity

at that location is changing with time (unsteady flow) (*Figure* I b). Call this acceleration a_t. This acceleration is like the acceleration of the disks in our first example or in the second example if the feed rate was changing with time.

The second way a fluid particle can accelerate is when it goes from a location where the velocity is one value to another location where the velocity is different; velocity changes along the length of the tube through which the particle is moving (*Figure* I a). Call this acceleration a_s. If the stream-tube is narrowing at some point, then the fluid particle will accelerate at that point; if the tube is widening then the fluid particle will decelerate. This acceleration is like the acceleration of the disks moving in the narrowing passage.

In general, acceleration will contain both terms a_t and a_s. The expression for acceleration along the flow direction is (see *Box* 1 for derivation):

$$a = a_t + a_s = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial S} = \frac{\partial U}{\partial t} + \frac{\partial(U^2/2)}{\partial S} \qquad (1)$$

Figure I. Change in velocity (ΔU) of a fluid particle as it moves from point 1 to point 2. The graphs below each figure indicate the variation of fluid velocity in the stream-tube. a) Velocity at a point does not change with time, but velocity varies along the stream-tube. $\Delta U = \Delta U_s$ b) Velocity is constant along the stream-tube, but varies with time. $\Delta U = \Delta U_t$. c) Velocity varies in space and time. $\Delta U = \Delta U_s + \Delta U_t$.
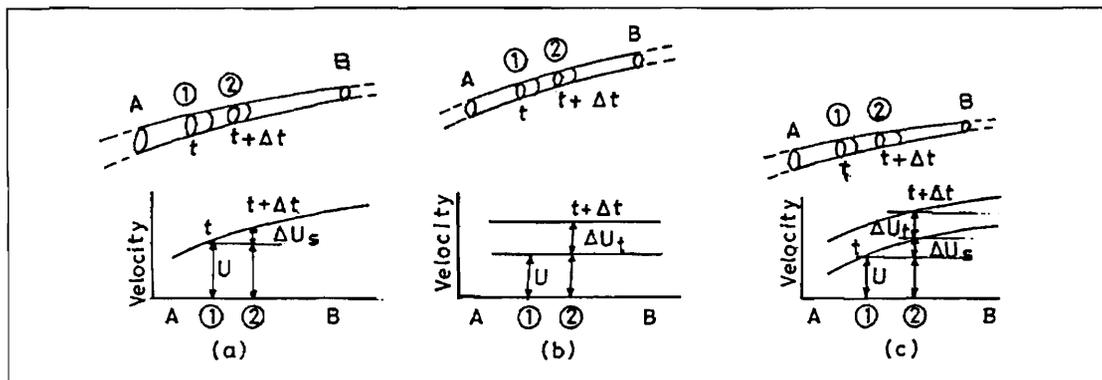
Box 1. Acceleration of a Fluid Particle

We derive the acceleration along the flow direction of a fluid particle moving in a thin stream tube. Coordinate along the stream-tube is S. Consider three cases: 1) Steady flow in a stream-tube whose cross-sectional area varies along the tube; fluid velocity changes along the tube but not with time. 2) Constant area stream-tube; velocity changes with time but is constant along the stream-tube. 3) Fluid velocity changes both along the tube and with time.

Case 1 is shown *Figure* Ia. The stream-tube extends from point A to point B, and the cross-sectional area continuously reduces from A to B. So the fluid velocity, shown schematically in the figure, increases from A to B. Now consider a fluid particle which at time $= t$ is at point 1 and its velocity is U ; a short time, $t+\Delta t$, later the the fluid particle moves a distance $\Delta S = U\,\Delta t$ to point 2 in the tube. We come to a most important point: since the particle has moved to a new position, its velocity changes to a value, $(U + \Delta U_s$), corresponding to the new position 2 (see *Figure* Ia). (Subscript s denotes case when changes are due to change in particle position.) In the example we are considering the stream-tube narrows and velocity increases ($\Delta U_s > 0$); if the stream-tube were to widen then $\Delta U_s < 0$. Acceleration in the flow direction is given by $a_s \simeq \Delta U_s\,/\,\Delta t$. Writing $\Delta t \simeq \Delta S/U$, we get $a_s = \mathrm{U}\Delta U_s/\,\Delta S$. In the limit $\Delta t \to 0$, we get

$$a_s = U\frac{\partial U}{\partial S} = \frac{\partial U^2/2}{\partial S}.$$

In Case 2 (*Figure* Ib) fluid velocity changes with time but not along the stream-tube. Consider again the motion of a fluid particle from point 1 to point 2 in time Δt. At the initial time all the fluid particles in the stream-tube have the same velocity $= U$, and at the later time all the fluid particles have velocity $= U + \Delta U_t$ (see *Figure* I b). (Subscript t denotes case when changes are due to changes in time.) Acceleration is thus given by $a_t \simeq \Delta U_t/\,\Delta t$, and in the limit $\Delta t \to 0$ we get

$$a_t = \frac{\partial U}{\partial t}.$$

In general fluid velocity can change both along the tube and with time (Case 3). The fluid velocity is shown schematically in *Figure* Ic. ΔU_s and ΔU_t are non-zero. Fluid acceleration is the sum of a_t and a_s and is

$$a = a_t + a_s = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial S} = \frac{\partial U}{\partial t} + \frac{\partial U^2/2}{\partial S}.$$

The term a_s is called convective acceleration. Note that it is the product of how fast the particle is moving (U) and how fast the velocity is changing in space $(\frac{\partial U}{\partial S})$. It is something like when climbing a hill, the rate at which you gain altitude depends on how fast you walk and how steep the slope is. The above acceleration is along the stream-tube or flow direction. When a fluid particle is moving along a curved path there is also a component of acceleration component normal to the flow direction – the centripetal acceleration. The acceleration along the stream-tube is due to change in magnitude of the velocity of the fluid particle. Centripetal acceleration is due to change in direction of the velocity. For Bernoulli's equation we are concerned only with acceleration along the flow direction.
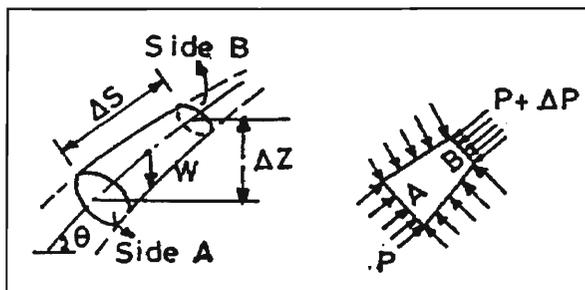
Forces on a Fluid Particle

Usually three types of forces are present. One is the weight of the fluid particle. The other two are pressure and viscous forces, both molecular in origin. Again considering the case of a gas, pressure is caused by 'bombarding' of the molecules on a surface. In analogy with the motion of disks we considered earlier, pressure is like the normal force between discs. Viscous force is like the friction force between adjacent disks. However, unlike in the disks case where friction can exist even when there is no motion, in a fluid viscous force is present only when there is relative motion between adjacent particles and is proportional to how fast these particle slide past each other. Bernoulli's equation is derived considering only the forces due to pressure and gravity.

Consider a small fluid particle of length ΔS, cross-sectional area A, and density ρ (*Figure* J). Its mass $m = \rho A \Delta S$. Z direction points upwards, and flow direction makes an angle θ with the horizontal plane. The force on a fluid particle in the flow direction due to its weight is (see *Box* 2).

**Figure J. Forces on a fluid
particle: weight (W) and
force due to pressure dif-
ference ΔP.**

$$F_G = -mg\sin\theta = -\rho g A \Delta S \frac{dZ}{dS}. \qquad (2)$$

Pressures on the two sides of the particle are different by ΔP (*Figure* J). The force in the flow direction due to this pressure difference is (see *Box* 2).

Box 2. Force on a Fluid Particle

Consider a very small (infinitesmal) fluid particle, length $= \Delta S$, that is within a stream-tube (*Figure* J). The flow direction makes an angle θ with the horizontal plane. Side A of the fluid particle has area A and on it the pressure is P; side B has area $A + \Delta A$ and on it the pressure is $P + \Delta P$. The length of the fluid particle is ΔS, and the elevation from A to B changes by ΔZ. $\frac{dZ}{ds} = \sin\theta$.

The volume of the fluid particle $\simeq A\ \Delta S$, and its weight is $\rho g A\ \Delta S$, where ρ is fluid density and g is acceleration due to gravity. Thus the component of the weight acting in the flow direction S is

$$F_G \simeq -\rho g A \Delta S \sin\theta = -\rho g A \Delta Z.$$

The force in the flow direction on the fluid particle due to pressure is

$$F_P = -(P + \Delta P)(A + \Delta A) + PA +\ \text{force due to pressure on side wall.}$$

For small ΔS it can be shown force due to pressure on side wall $= -P\Delta A$. Therefore

$$F_P = -A\Delta P = -A\Delta S \frac{\partial P}{\partial S}.$$

Neglecting viscous forces, the total force on the fluid particle $= F_P + F_G$.

$$F_P = -A\Delta P = -A\Delta S \frac{\partial P}{\partial S}.$$ (3)

Pressure increasing with S tries to slow the particle down, and pressure decreasing with S tries to accelerate the particle.

Bernoulli's Equation

Neglecting viscous forces (ideal flow) we have pressure force and weight producing the acceleration of the fluid particle: $a_t + a_s = (F_G + F_P)/m$. Substituting for the terms using (1),(2) and (3) and taking the limit of the particle size going to zero, we get the differential equation of motion along a streamline

$$\frac{\partial U}{\partial t} + \frac{\partial (U^2)/2}{\partial S} = -\frac{1}{\rho}\frac{\partial P}{\partial S} + g\frac{dZ}{dS}$$

Assuming density is a constant and integrating between two points 1 and 2 along a streamline we get the Bernoulli equation for unsteady, incompressible flow:

$$\int_1^2 \frac{\partial U}{\partial t}dS + \frac{U_2{}^2}{2} + \frac{P_2}{\rho} + gZ_2 = \frac{U_1{}^2}{2} + \frac{P_1}{\rho} + gZ_1.$$

U_1, P_1, and Z_1 are respectively the velocity, pressure and elevation at point 1 on the streamline, and U_2, P_2, and Z_2 are respectively the velocity, pressure and elevation at point 2 on the streamline. The first term on the left hand side is related to change of velocity with time of all the fluid particles lying in the stream-tube. The velocity terms ($U_2{}^2$ and $U_1{}^2$) are related to change in velocity along the streamtube. These two velocity changes are caused by pressure and gravitational forces. By integrating we are summing the $F = ma$ equation over the string of particles in the streamtube (*Figure* K) (as we did in the case of the row of disks) and the result is Bernoulli's equation.
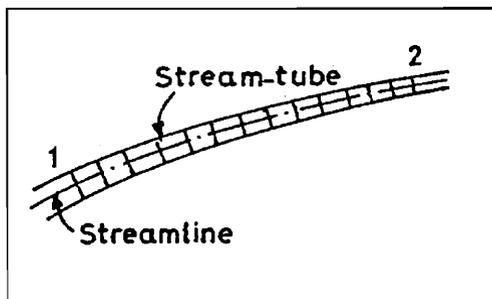
For steady flow the first term on the left hand side is zero and we get the familiar and elegant Bernoulli equation:

$$\frac{U_2^2}{2g} + \frac{P_2}{\rho g} + Z_2 = \frac{U_1^2}{2g} + \frac{P_1}{\rho g} + Z_1.$$

Written this way all the terms have dimension of length. The three terms on either side of the equation are velocity head, pressure head and gravity head. Another form, frequently used, is obtained by multiplying the equation through by ρg; then all the terms will have dimensions of pressure.

Points 1 and 2 can be anywhere on the streamline. Thus along the streamline we have in steady flow

$$\frac{U^2}{2g} + \frac{P}{\rho g} + Z = \text{constant} = H = \frac{P_0}{\rho g}. \qquad (4)$$

H is called the Bernoulli head and is a constant along any streamline. P_0 is the stagnation pressure. The values of H and P_0 can be different for different streamlines. In a type of flow which is frequently encountered, called irrotational flow, the Bernoulli head is the same for all streamlines. Then we can apply the above equation between any two points in the flow; the two points need not lie on the same streamline.

The Bernoulli equation can also be viewed in terms of energy: work done = change in energy, we assume there is no heat transfer. We look at the case when the fluid

density can change, fluid is compressible. The rate at which work is done on a fluid particle is net force in the flow direction on the fluid particle multiplied by its velocity. In the absence of viscous forces, only forces due to pressure can do work. This work not only changes the kinetic energy and gravitational potential energy, but also the internal energy (proportional to temperature) of the particle. The internal energy changes through PdV (pressure times change in volume) type of work: the contraction or expansion of a particle is caused by change in pressure as the particle moves from one location to another. The energy equation or the Bernoulli equation for steady compressible flow (see *Box* 3) is

$$\frac{U_2{}^2}{2} + \frac{P_2}{\rho_2} + gZ_2 + e_2 = \frac{U_1{}^2}{2} + \frac{P_1}{\rho_1} + gZ_1 + e_1,$$

where e is the internal energy and points 1 and 2 lie on the same streamline.

The compressible version of Bernoulli's equation is used for flow of gases at high speeds (speeds greater than about 0.3 times the speed of sound in that gas). Under these conditions the gravitational term is negligible and also it is usual to write the equation in terms of enthalpy, $h = e + p/\rho$. Then the 'compressible' Bernoulli equation becomes

$$h + \frac{U^2}{2} = \text{constant along a streamline} = h_0,$$

which for a perfect gas can be written as

$$C_p T + \frac{U^2}{2} = C_p T_0,$$

where C_p is the specific heat at constant pressure. h_0 and T_0 are stagnation enthalpy and stagnation temperature respectively, values which would be obtained at a point where the velocity is zero.

Figure K. A fluid mass at time t is in A and B. At time $(t+\Delta_t)$ the fluid mass has moved and is in C and D. The change in energy of the fluid mass in the time interval Δ_t is equal to the work done during this time.
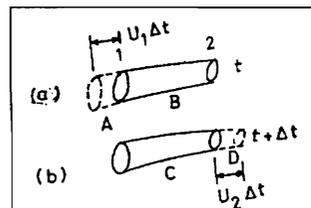
Box 3. Energy Equation

Consider steady frictionless flow in a stream-tube lying between points 1 and 2 (*Figure* L). Fluid density is allowed to change. We apply the energy conservation equation (first law of thermodynamics) to a fluid volume (system). At the initial time t the fluid system is divided into two parts, A and B. Part A is outside the stream-tube, and B completely fills the stream-tube. At a small time Δt later fluid A has entered the stream-tube and fluid D which was inside the tube (and part of B) has come out; at this time we divide the system into C and D, with C filling the tube. The work done in time Δt on the system is entirely due to pressure at the two ends of the stream-tube, and is

$$(P_1 A_1) U_1 \Delta t - (P_2 A_2) U_2 \Delta t, \tag{1}$$

where P_1, A_1 and U_1 are respectively pressure, cross-sectional area and velocity at end 1 of the streamtube, and P_2, A_2 and U_2 are respectively pressure, cross-sectional area and velocity at end 2. In the case of steady flow the mass entering the streamtube = mass leaving the streamtube, i.e., mass in A (m_A) = mass in D (m_D), or

$$\rho_1 A_1 U_1 \Delta t = \rho_2 A_2 U_2 \Delta t, \tag{2}$$

where ρ_1 and ρ_2 are respectively densities at ends 1 and 2 of the streamtube. Cancelling Δt on the two sides we get the mass conservation equation for steady flow, $\rho_1 A_1 U_1 = \rho_2 A_2 U_2$, which for incompressible flow $(\rho_1 = \rho_2)$ becomes $A_1 U_1 = A_2 U_2$ (velocity times cross-sectional area is constant). Using (2) the work done (1) can be written as $m_A(P_1/\rho_1 - P_2/\rho_2)$. The work done is equal to the energy gained by the system in time Δt, i.e.,$(E_C + E_D - E_A - E_B)$, where E is energy and the subscripts refer to different parts A, B, etc. The steady state assumption implies $E_C = E_B$. Energy consists of kinetic energy, gravitational potential energy and internal energy. Thus $E_A = m_A(1/2 U_1^2 + g Z_1 + e_1)$ and $E_D = m_D(1/2 U_2^2 + g Z_2 + e_2)$, where e is the internal energy per unit mass. Equating work done to energy change gives

$$\frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} = 1/2 U_2^2 - 1/2 U_1^2 + g(Z_2 - Z_1) + e_2 - e_1. \tag{3}$$

Since points 1 and 2 can be anywhere on the streamline we can write

$$P/\rho + 1/2 U^2 + gZ + e = \text{ constant on a streamline}, \tag{4}$$

which is the energy equation for steady frictionless compressible flow or the Bernoulli equation for compressible flow. The above equation has been derived by applying the energy equation to a fluid volume in the stream-tube. The same equation can be derived by applying the energy equation to a fluid particle as it moves from points 1 to 2.

When a fluid is incompressible then the volume of the element cannot change, there is no change in internal energy (in the absence of viscous stresses), the work done by pressure changes only the kinetic and potential energies and we recover the earlier Bernoulli equation (4).

In any use of Bernoulli's equation the main assumption – negligible viscous forces – should be kept in mind.

Conclusion

In a forthcoming article we will look at some examples of the application of Bernoulli's equation. From this article I hope the reader has developed a feel for some aspects of fluid motion: the concept of a fluid particle, the two types of fluid acceleration and how motion in one part of the fluid causes motion in other parts of the fluid. Bernoulli's equation can be viewed in two ways. One as Newton's second law applied to a line of fluid particles in a stream-tube. The second as a statement of energy conservation: the change in gravitational potential energy plus the change in kinetic energy is equal to the work done by the pressure forces.

Finally, one can observe fluid motion in small streams, or on a roadside after rain, aided by small leaves or other floating debris which are carried by the flow and act as tracers. A leaf accelerates when a passage through which the water is flowing narrows; or it 'senses' and changes direction when it approaches an obstacle like a small stone. Sometimes one sees a thin layer of oil on top of the water and then the relative motion or distortion of the flowing fluid can be observed. These tracers simply show what the fluid particles below them are doing.

Address for Correspondence
Jaywant H Arakeri
Department of Mechanical
Engineering
Indian Institute of Science
Bangalore 560 012, India.

Number rules the universe.

Pythagorus