Triangularization of a Matrix

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Two ideas that pervade all of mathematics are *equivalence*, and the related notion of *reduction.* If an object in a given class can be carried into another by a transformation of a special kind, we say the two objects are equivalent. Reduction means the transformation of the object into an equivalent one with a special form as simple as possible.

The group of transformations varies with the problem under study. In linear algebra, we consider arbitrary non-singular linear transformations while studying algebraic questions. In problems of geometry and analysis, where distances are preserved, unitary (orthogonal) transformations alone are admitted. In several problems of crystallography and number theory, the interest is in linear transformation with integral coefficients and determinant one.

In this article, we restrict ourselves to $n \times n$ complex matrices. Two such matrices A and B are said to be *similar* if there exists a non-singular (invertible) matrix S such that $B = S^{-1}AS$. If this S can be chosen to be unitary ($S^{-1} = S^{\star}$) we say that A and B are *unitarily similar.* Similar matrices are representations of the same linear operator on $\mathbb{C}^n$ in two different bases. Unitarily similar matrices represent the same linear operator but in two different *orthonormal* bases. Similarity and unitary similarity are equivalence relations.

Similarity preserves (does not change) the rank, determinant, trace and eigenvalues of a matrix. Unitary similarity preserves all these and more. For example if A is Hermitian ($A = A^{\star}$), then every matrix unitarily similar to it is Hermitian too. If we define the *norm* of any matrix A as

$$\| A \|_2 = (\sum_{i,j} |a_{ij}|^2)^{1/2}$$

then every matrix unitarily similar to A has the same norm. The simplest way to see this is to note that

$$\| A \|_2 = (\mathrm{tr} A^\star A)^{1/2} = \| U^\star A U \|_2,$$

where tr stands for the trace of a matrix.

It is generally agreed that the more zero entries a matrix has, the simpler it is. Much of linear algebra is devoted to reducing a matrix (via similarity or unitary similarity) to another that has lots of zeros.

The simplest such theorem is the *Schur triangularization theorem.* This says that *every matrix is unitarily similar to an upper triangular matrix.*

Our aim here is to show that though it is very easy to prove it, this theorem has many interesting consequences.

Proof of Schur's Theorem

We want to show that given an $n \times n$ matrix A, there exists a unitary matrix U and an upper triangular matrix T such that $A = UTU^\star$ This is equivalent to saying that there exists an orthonormal basis for $\mathbb{C}^n$ with respect to which the matrix of the linear operator A is upper triangular. In other words, there exists an orthonormal basis $v_1, \quad , v_n$ such that for each $k = 1, 2, \quad . n$, the vector Av_k is a linear combination of $v_1, \quad , v_k$.

This can be proved by induction on n. Let λ_1 be an eigenvalue of A and v_1 an eigenvector of norm one corresponding to it. Let M be the one-dimensional subspace of $\mathbb{C}^n$ spanned by v_1, and let N be its orthogonal complement. Let P_N be the orthogonal projection with range N For $y \in N$, let $A_N y = P_N A y$. Then A_N is a linear operator on the $(n-1)$-dimensional space N

By the induction hypothesis, there exists an orthonormal basis $v_2, \quad, v_n$ of N such that the vector $A_N v_k$ for $k = 2, \quad. n$ is a linear combination of $v_2, \quad, v_k$. The set $v_1, \quad, v_n$ is an orthonormal basis for $\mathbb{C}^n$ and each $Av_k, 1 \leq k \leq n$, is a linear combination of $v_1, \quad, v_k$. This proves the theorem. The basis $v_1, \quad, v_n$ is called a *Schur basis* for A.

Notice that we started our argument by choosing an eigenvalue and eigenvector of A. Here we have used the fact that we are considering complex matrices only. The diagonal entries of the upper triangular matrix T are the eigenvalues of A. Hence, they are uniquely specified up to permutation. The entries of T above the diagonal are not unique. Since,

$$\sum_{i,j} |t_{ij}|^2 = \sum_{i,j} |a_{ij}|^2$$

they cannot be too large. The reader should construct two 3×3 upper triangular matrices which are unitarily similar.

The Spectral Theorem

A matrix A is said to be *normal* if $AA^\star = A^\star A$. Hermitian and unitary matrices are normal.

The spectral theorem says that a *normal matrix is unitarily similar to a diagonal matrix.*

This is an easy consequence of Schur's theorem: Note that the property of being normal is preserved under unitary similarity, and check that an upper triangular matrix is normal if and only if it is diagonal.

The Schur basis for a normal matrix A is thus a basis consisting of eigenvectors of A. Normal matrices are, therefore, matrices whose eigenvectors form an orthonormal basis for $\mathbb{C}^n$

Some Density Theorems

A subset Y of a metric space X is said to be *dense* if every neighbourhood of a point in X contains a point of Y This is equivalent to saying that every point in X is the limit of a sequence of points in Y (The set of rational numbers and the set of irrational numbers are dense in $\mathbb{R}$.)

The space $\mathbb{M}(n)$ consisting of $n \times n$ matrices is a metric space if we define for every pair A, B the distance between them as $d(A,B) = \| A - B \|_2$. We will show that certain subsets are dense in $\mathbb{M}(n)$. The argument in each case will have some common ingredients. The property that characterizes the subset Y in question will be one that does not change under unitary similarity. So, if $A = UTU^\star$ and we show the existence of an element of Y in an ϵ-neighbourhood of an upper triangular T, then we would have also shown the existence of an element of Y in an ϵ-neighbourhood of A.

Invertible matrices are dense. A matrix is invertible if and only if it does not have zero as an eigenvalue. This property is not affected by unitary similarity. We want to show that if A is any matrix then for every $\epsilon > 0$, there exists an invertible matrix B such that

$\| A - B \|_2 < \epsilon$. Let $A = UTU^\star$,

where T is upper triangular. If A is singular some of the diagonal entries of T are zero. Replace them by small non-zero numbers so that for the new upper triangular matrix T' obtained after these replacements, we have $\| T - T' \|_2 < \epsilon$. Then T' is invertible and so is $A' = UT'U^\star$. Further,

$$\| A - A' \|_2 = \| U(T - T')U^\star \|_2 < \epsilon.$$

Matrices with distinct eigenvalues are dense. Use the same argument as above. If any two diagonal entries of T are equal, change one of them slightly.

Diagonalizable matrices are dense. A matrix is said to be *diagonalizable* if it is similar to a diagonal matrix; i.e. if it has n linearly independent eigenvectors. Since eigenvectors corresponding to distinct eigenvalues of any matrix are linearly independent, every matrix with distinct eigenvalues is diagonalizable, (the converse is not true). So the set of diagonalizable matrices includes a dense set (matrices with distinct eigenvalues) and hence is itself dense.

These density theorems are extremely useful. Often it is easy to prove a statement for invertible or diagonalizable matrices. Then one can extend it to all matrices by a limiting procedure. We give some examples of this argument.

The exponential of a matrix is defined as

$$e^A = I + A + \frac{A^2}{2!} +$$

(The series is convergent.) We want to calculate the determinant $\det(e^A)$. It turns out that $\det(e^A) = e^{\mathrm{tr}(A)}$. This is obviously true if A is a diagonal matrix: if the diagonal entries of A are $\lambda_1, \quad \lambda_n$ then $\det(e^A) = e^{\lambda_1}$ $e^{\lambda_n} = e^{\lambda_1 + \ldots + \lambda_n} = e^{\mathrm{tr}(A)}$. From this one can see that this equality is also true for diagonalizable matrices; just note that $e^{SAS^{-1}} = Se^A S^{-1}$. Finally, the equality carries over to all matrices since both sides are continuous functions of a matrix and every matrix is a limit of diagonalizable matrices.

Let A, B be any two matrices. We know that $\det(AB) = \det(BA)$, and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. More generally, it is true that AB and BA have the same characteristic polynomial and hence the same eigenvalues (including multiplicities). Recall that the k-th coefficient in the characteristic polynomial of A is (up to a sign) the sum of $k \times k$ principal minors of A. These are polynomial functions of the entries of A, and hence depend continuously on

A. Thus, to prove that AB and BA have the same characteristic polynomial, it is enough to prove this when B belongs to a dense subset of $\mathbb{M}(n)$. The set of invertible matrices is such a set. But if B is invertible, then $B(AB)B^{-1} = BA$, i.e. AB and BA are similar. Hence, they have the same characteristic polynomial.

This theorem, in turn, is very useful in several contexts. Let A and B be two positive semidefinite matrices. Then all their eigenvalues are non-negative. The product AB is not Hermitian (unless A and B commute), so *a priori* it is not even clear whether AB has real eigenvalues. We can, in fact, prove that it has non-negative real eigenvalues. Let $B^{1/2}$ be the unique positive square root of B. Then $AB = (AB^{1/2})B^{1/2}$ and this has the same eigenvalues as $B^{1/2}AB^{1/2}$. This matrix is positive semidefinite, and hence has non-negative eigenvalues.

The *Cayley Hamilton theorem* says that every matrix satisfies its characteristic equation; i.e. if $\chi(z)$ is the polynomial in the variable z obtained by expanding $\det(zI - A)$, and $\chi(A)$ is the matrix obtained from this polynomial on replacing z by A, then $\chi(A) = 0$. The reader is invited to write a proof for this using the above ideas; the proof is easy for diagonal matrices.

A Bound for Eigenvalues

In many problems it is of interest to calculate the eigenvalues of a matrix A. This is not always easy. Sometimes, it helps to know the eigenvalues approximately, or at least that they lie (or do not lie) in some region of the complex plane. From Schur's theorem, it is clear that, if λ_i are the eigenvalues of A, then

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i,j} |a_{ij}|^2.$$

The two sides are equal if and only if A is normal.

This leads to an amusing (but not the easiest) proof of the arithmetic-geometric mean inequality. Let $a_1, \quad a_n$ be non-negative numbers. The eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & a_1 & 0 & & 0 \\ 0 & 0 & a_2 & & 0 \\ \vdots & & & & \\ 0 & 0 & & & a_{n-1} \\ a_n & 0 & & & 0 \end{pmatrix}$$

are the n-th roots of $a_1a_2 . \quad a_n$. Hence by the above inequality

$$n(a_1a_2 \quad a_n)^{2/n} \leq {a_1}^2 + \quad + {a_n}^2.$$

Changing ${a_i}^2$ to a_i, we get the inequality

$$(a_1a_2 \quad a_n)^{1/n} \leq \frac{a_1 + \quad + a_n}{n}$$

between the geometric mean and the arithmetic mean. We even get the condition for equality; just note that A is normal if and only if $a_1 = a_2 = \quad = a_n$.

Here is a more serious and powerful application of these ideas.

Theorem. If A, B are normal matrices such that AB is normal, then BA is also normal.

Proof. Let $\lambda_i(AB), 1 \leq i \leq n$, be the eigenvalues of AB. Since AB is normal

$$\sum_{i=1}^{n} |\lambda_i(AB)|^2 = {\parallel AB \parallel_2}^2$$

To prove that BA is normal, we have to show that this is true when AB is replaced by BA. We have seen that $\lambda_i(AB) = \lambda_i(BA)$. So, we have to show that

$${\parallel AB \parallel_2}^2 = {\parallel BA \parallel_2}^2$$

i.e.,

$$\mathrm{tr}(B^\star A^\star AB) = \mathrm{tr}(A^\star B^\star BA).$$

Using the fact that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for all matrices X Y and the normality of A, B, the two sides of this desired equality are seen to be equal to $\mathrm{tr}(AA^\star BB^\star)$. This proves the theorem.

The reader might try to find another proof of this theorem. (If the reader is unable to find such a proof from the mere definition of normality, she should not be surprised. The statement is false in infinite-dimensional Hilbert spaces. It is, however, true if one of the operators A or B is compact.)

Commuting Matrices

Let A and B be two matrices. Schur's theorem tells us that there exist unitary matrices U, V and upper triangular matrices R, T such that $A = URU^\star, B = VTV^\star$. It turns out that if A and B commute ($AB = BA$), then we can choose $U = V$ In other words, if A and B commute, they have a common Schur basis.

To prove this, we first show that A, B have a common eigenvector. Let λ be an eigenvalue of A, and let $W = \{x : Ax = \lambda x\}$ be the associated eigenspace. If $x \in W$ then

$$ABx = B(Ax) = B(\lambda x) = \lambda(Bx).$$

Thus, $Bx \in W$ This says that the space W is *invariant* under B. So, there exists $y \in W$ such that $By = \mu y$. This y is a common eigenvector for A and B.

The rest of the proof is similar to the one we gave earlier for Schur's theorem.

The same argument shows that if $\{A_\alpha\}$ is any family of pairwise commuting matrices, then all A_α have a common Schur basis.

Distance between Eigenvalues

Let A and B be commuting matrices with eigenvalues $\lambda_1, \quad, \lambda_n$ and $\mu_1, \quad, \mu_n$, respectively. We have seen that there exists a unitary matrix U such that $A = UTU^\star, B = UT'U^\star$. The diagonal entries of T and T' are the numbers $\lambda_1, \quad \lambda_n$ and $\mu_1, \quad, \mu_n$ (in some order). Hence,

$$(\sum_{i=1}^{n} |\lambda_i - \mu_i|^2)^{1/2} \leq \| T - T' \|_2 \leq \| A - B \|_2$$

Thus, it is possible to enumerate the n-tuples $\{\lambda_j\}$ and $\{\mu_j\}$ so that the *distance between them is smaller than the distance between A and B* (in the sense made precise by this inequality).

This is no longer true if A and B do not commute. For example, consider

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$$

A famous theorem of Hoffman and Wielandt says that if A and B are both normal, then the above inequality is true even when A, B do not commute.

This article is based on a talk given by the first author at a refresher course for college teachers organized by CPDHE in April 1999.

Suggested Reading

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