Think It Over

This section of Resonance presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', Resonance, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

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The Limit of n-fold Sine

Consider the function $f_n(x)$ defined for positive integers $n$ and real numbers $x$ by $f_n(x) = \sin(\sin(\cdots(\sin(x))\cdots))$, with $n$ applications of the sine function. Thus $f_2(x) = \sin(\sin(x))$, $f_3(x) = \sin(\sin(\sin(x)))$, and so on.

Let $x \in (0, \pi/2)$ be chosen, and let the sequence

$$f_1(x), f_2(x), f_3(x), \ldots, f_{100}(x), \ldots$$

be computed. Naturally the sequence depends on $x$, but curiously as $n \to \infty$ the dependence seems to diminish. Indeed we find that for $n=4000$,

$$f_n(0.3) = 0.027603, f_n(0.5) = 0.027333, f_n(1) = 0.027361,$$

while for $n=10000$,

$$f_n(0.3) = 0.017289, f_n(0.5) = 0.017307, f_n(1) = 0.017314$$

(all values correct to 5 decimal places). In fact it turns out that for fixed $x$ and sufficiently large $n$, we have $f_n(x) \approx \sqrt{3/n}$

How may this phenomenon be explained?