Some Aspects of Fixed Point Theory[1]

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Introduction

Most of mathematics is focussed on the solution of various equations involving numbers. For example, for given real numbers a and b with $a \neq 0$ the linear equation $ax+b=0$ has a unique real solution. On the other hand the quadratic equation $ax^2+bx+c=0$ may not have real solutions for real numbers a, b and c with $a \neq 0$. However, it will always have a pair of solutions in the system of complex numbers. More generally, one can consider an equation of the form $g(x) = 0$, where g is a real-valued function of a real variable. For $f : R \to R$ defined by $f(x) = g(x)+x$, obviously a solution of $g(x) = 0$ is a solution of $f(x) = x$ and conversely. An element x_0 for which $f(x_0) = x_0$ is called a fixed-point of f. Thus the problem of solving the equation $g(x) = 0$ is equivalent to finding the fixed points of an associated function f. On the face of it, it may appear strange that in order to find the zeros of a function g, one should seek the fixed points of a related function such as f. Nevertheless this approach has been found to be effective in solving many nonlinear equations.

Fixed-point Theorems

We begin with

Definition. Let $f : X \to X$ be a map where X is any non-void set. An element x_0 with $f(x_0) = x_0$ is called a fixed-point of f.

A fixed-point theorem is one which ensures the existence of a fixed point of a mapping f under suitable assumptions both on X and f. As has been already pointed out, many nonlinear equations can be solved using fixed

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point theorems. In fact fixed point theorems find applications in nonlinear integral, differential equations, game theory and optimization theory.

Apart from establishing the existence of a fixed point, it often becomes necessary to prove the uniqueness of the fixed point. Besides, from a computational point of view, an algorithm for calculating the value of the fixed point to a given degree of accuracy is desirable. Often this algorithm involves the iterates of the given function. In essence, the questions about the existence, uniqueness and approximation of a fixed point provide three significant aspects of the general fixed point principle. Banach's contraction principle which is well-known to the students of mathematical analysis is perhaps one of the few most significant theorems. Not only is its proof elementary, but it also answers all the three questions of existence, uniqueness and constructive algorithm convincingly.

A deeper, though special result, is Brouwer's fixed point theorem. It states that any continuous function mapping a closed ball $\overline{B(a,r)}$ of R^n into itself has a fixed point. In general, Brouwer's theorem ensures neither the uniqueness of the fixed point nor the convergence of the iterates. While the early proofs of Brouwer's theorem rely on algebraic-topological ideas, proofs based purely on analytical arguments are available now.

Fixed Points and Maps on Compact Real Intervals

The proof of Brouwer's theorem in R^1 is elementary and hardly can one resist the temptation to provide it here as well. It is readily seen that in R^1 the closed balls are precisely the closed bounded intervals of real numbers. Thus we have

Theorem 1. *If $f : [a,b] \to [a,b]$ is continuous then f has a fixed point.*

Proof. If $f(a) = a$ or $f(b) = b$, then the conclusion is obvious. Otherwise $a < f(a)$ and $f(b) < b$. Define the continuous function $g : [a, b] \to R$ by $g(x) = f(x) - x$. Clearly $g(a) > 0$, while $g(b) < 0$. Thus the continuous function g changes sign on $[a, b]$. So by the Intermediate Value Property for continuous functions, g has a zero in $[a, b]$. This zero is a fixed point of f. Q.E.D.

Several questions arise. By considering the mapping $x \to x + 1$ on $[0, \infty]$, it is clear that continuous self-maps on closed unbounded intervals of R may not have fixed points. The mapping $x \to \frac{x+1}{2}$ on $(0, 1)$, though continuous, has no fixed point in $(0, 1)$. So the assumption that the interval is closed cannot be weakened in the theorem.

It is natural to explore if the above theorem remains true even for certain mappings that may not be continuous. More specifically we ask whether any monotonic nondecreasing function mapping $[a, b]$ into itself has a fixed point. The answer to this question is in the affirmative. Indeed a more general theorem due to Tarski is true. This will be discussed in the following section.

Tarski's Fixed Point Theorem

If we examine any closed and bounded interval I of R such as $[0, 1]$ it is clear that any nonempty subset of this set I has a supremum and infimum in I In fact one can consider more general structures called lattices, defined below.

Definition. A partially ordered set L is called a lattice if for any pair $a, b \in L$, infimum $\{a, b\}$ and supremum $\{a, b\}$ also belong to L.

A lattice L is said to be complete if it contains the supermum and infimum of each of its nonempty subsets.

Definition. A mapping $f : L \to L$ where L is a partially ordered set with partial order $\leq$, is called isotone if $f(x) \leq f(y)$ for all $x, y \in L$ with $x \leq y$.

There are a number of examples of complete lattices.

Examples.

1. $[a, b]$ with the usual order is a complete lattice. In fact it is totally ordered and complete.

2. Define a partial order $\leq$ on $I = [a, b] \times [c, d]$ by $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$. Under this order $[a, b] \times [c, d]$ is a complete lattice which is not totally ordered. In fact given $(x_1, y_1), (x_2, y_2) \in I$ none of these elements need be infimum or supremum of the pair. More specifically, given $(0, 1)$ and $(1, 0) \in [0, 1] \times [0, 1]$, $\inf \{(0, 1), (1, 0)\} = (0, 0)$ whereas $\sup \{(0, 1), (1, 0)\} = (1, 1)$. (Try to prove this!)

3. On $C[a, b]$, the space of real-valued continuous functions on $[a, b]$, define a partial order by $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in [a, b]$. Clearly $C[a, b]$ is a lattice. It is not complete. For example the infimum of the set $S = \{f_n : n \in N\}$ where $f_n(x) = x^n$ in $C[0, 1]$ does not belong to $C[0, 1]$.

4. In $L_p[a, b], p \geq 1$ the space of Lebesgue measurable real-valued functions on $[a, b]$ for which the p-th power is integrable, a partial order $\leq$ is defined by: $f \leq g$ if and only if $f(x) \leq g(x)$ for almost all $x \in [a, b]$. Then

$$A = \{f \in L_p[a, b] : 0 \leq f \leq 1\}$$

is a complete lattice (Prove this!).

5. For any nonempty set X, the power set $P(X)$ of X, viz., the set of all subsets of X, is a complete lattice with respect to set-inclusion.

Tarski [2] proved the following fixed point theorem which he obtained already in 1939 (see footnote 1 in [2]).

Theorem. (Tarski). Let $f : A \to A$ *be an isotone* *function on the complete lattice* $(A, \leq)$. *Then* P, *the set of all fixed points of* f, *is a nonempty complete sub-lattice. Further,* $\sup\{x \in A : f(x) \geq x\}$ *and* $\inf\{x \in A : f(x) \leq x\}$ *are also fixed points of* f.

Proof. Let $0 = \inf A$. Then $f(0) \geq 0$. So the set $S = \{x \in A : f(x) \geq x\}$ contains 0 and is nonempty. So S has a supremum in A. Let $u = \sup S$. Since $x \leq u$ for each x in $S, f(x) \leq f(u)$ by isotonicity of f. But for $x \in S, x \leq f(x)$. So, $x \leq f(x) \leq f(u)$ for all $x \in S$. Thus $f(u)$ is an upperbound for S. Hence $u \leq f(u)$. By isotonicity of $f, f(u) \leq f(f(u))$. In view of this $f(u) \in S$. Since u is supremum of $S, f(u) \leq u$. So $u = f(u)$ and $P \neq \Phi$.

Given the partial order $\leq$ on A, we can define another partial order $\leq'$ called the opposite of $\leq$, by $x \leq' y$ if and only if $y \leq x$. It is clear that A with the opposite order is also a complete lattice and f is also isotone with respect to this order. So, by applying the above argument to $(A, \leq')$ we have $\sup\{x : f(x) \geq' x\} = \inf\{x : f(x) \leq x\} \in P$.

Let Y be a nonvoid subset of P and $y_0 = \sup Y$. For any $y \in Y, y \leq y_0$ and as f is isotone, we have $y = f(y) \leq f(y_0)$. Thus $f(y_0)$ is an upper bound for Y and $y_0 \leq f(y_0)$. Also, $y_0 \leq z$ implies $y_0 \leq f(y_0) \leq f(z)$. This shows that f maps the ordered interval $[y_0, 1]$ into itself, where $1 = \sup A$ and $[y_0, 1] = \{z \in A : y_0 \leq z \leq 1\}$. A routine verification shows that $[y_0, a]$ is a complete lattice with the induced ordering. Now applying the first part of the proof to this lattice, we conclude that $v = \inf\{z \in [y_0, 1] : f(z) \leq z\}$ is a fixed point of f. Clearly, $v \in P$ and $v \leq z$ for all $z \in [y_0, 1]$ with $f(z) \leq z$. We now proceed to show that v is the supremum of Y in P. By definition of $v, y_0 \leq v$. Thus, v is an upper bound for Y in P. Let w be any upper bound for Y in P. So, $y \leq w$ for all $y \in Y$ So, $y_0 = \sup Y \leq w$. Therefore $w \in [y_0, 1]$ and $f(w) = w$. From the definition of v, it follows that $v \leq w$ and hence v is the least upper bound of Y in P. Similarly, any non-void subset of P has a greatest lower bound. (This can be proved by considering the opposite order on P and using the just concluded argument.) Incidentally, this argument also proves that P is a lattice in its own right for we just have to take for Y a two point subset of P.

In view of Tarski's theorem any monotonic non-decreasing function from $[a, b]$ into itself has a fixed point, since $[a, b]$ is a complete lattice. Thus we have

Corollary. *If $f : [a, b] \to [a, b]$ is a monotonic non-decreasing function, then f has a fixed point.*

As a monotonic function is in general not continuous, the corollary supplements the theorem.

On the other hand a monotonic non-increasing function mapping a complete lattice into itself may not have a fixed point. This is illustrated by the following

Example 6. Define $f : [0, 1] \to [0, 1]$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0, 1/2) \\ 0 \text{ if } x \in [1/2, 1] \end{cases}$$

Clearly f is monotonic non-increasing on the complete lattice $[0, 1]$ with no fixed point.

Tarski [2] also proved the following theorem which ensures the existence of a common fixed point for a commuting family of isotone maps.

Theorems. *Let $(A, \leq)$ be a complete lattice and F be a commuting family of isotone functions mapping A into A. Then P, the set of common fixed points of all functions $f \in F$ is a nonempty complete sublattice of A. Further* $\sup\ \{x \in A : f(x) \geq x$ *for all* $f \in F\}$ *and* $\inf\{x \in A : f(x) \leq x,$ *for all* $f \in F\}$ *are common fixed points of F.*

Proof (outline). Let $u = \sup\{x \in A : f(x) \geq x$ for all $f \in F\}$. As in the proof of the theorem, $u \leq f(u)$ for all $f \in F$ and $g(u) \leq g(f(u)) = f(g(u))$ for any fixed $g \in F$ for all $f \in F$ So $g(u) \in S = \{x \in A : f(x) \geq x\}$ and so $g(u) \leq u$. g being an arbitrary function in $F, P \neq \Phi$.

The remaining part of the proof is entirely analogous to that of the theorem.

Applications of Tarski's Fixed Point Theorem

Tarski's theorem can be used to prove two classical theorems, vix., the Schröeder–Bernstein theorem and the Cantor–Bendixon theorem.

Theorem (Schröeder–Bernstein). If A and B are two sets, f is a one-to-one mapping from A into B and g is a one-to-one mapping from B into A, then there exists a bijection from A onto B.

If there is a one-to-one map from a set A into a set B then it is intuitively clear that there is a 'copy' of A in B and so B has at least as many elements as A. What Schröeder–Bernstein theorem asserts is this: if B has at least as many elements as A and A too has at least as many elements as B, then A and B have equal number of elements. A simple application of this theorem leads one to infer that $(-1, 1) \cup \{n + 1 : n \in N\}$ has the same cardinality as R.

The crucial step in the proof of the above theorem is the following result, often referred to as Banach's lemma. Its proof is based on Tarski's theorem.

Lemma. (Banach). If $f : A \to B$ and $g : B \to A$ are one-to-one functions, then there exists a subset S of A with $g(B - f(S)) = A - S$.

Proof. Let $P(A)$ and $P(B)$ denote the power sets of A and B respectively. Order $P(A)$ and $P(B)$ by set-inclusion, denoted by $\leq$ Define T: $P(A) \to P(A)$ by $T(S) = A - g(B - f(S))$. For $A_1, A_2 \leq A$ and $A_1 \leq$

$A_2, f(A_1) \leq f(A_2)$ and $B - f(A_1) \geq B - f(A_2)$. So $g(B - f(A_1)) \geq g(B - f(A_2))$ where $T(A_1) \leq A - g(B - f(A_2)) = T(A_2)$. Thus T is an isotone operator on the complete lattice $P(A)$. So by Tarski's theorem T has a fixed point S in $P(A)$. Thus $A - g(B - f(S)) = S$. Equivalently $A - S = g(B - f(S))$.

Now the proof of the Schröeder–Bernstein theorem follows easily.

Proof of Theorem. By Banach's lemma, there exists a subset of S of A with $A - S = g(B - f(S))$. Define $h : A \to B$ as follows:

$$h(x) = \begin{cases} f(x), x \in S \\ g^{-1}(x), x \in A - S \end{cases}$$

Clearly h is a mapping of A onto B and is 1 1. $h(A) = h(S \cup (A - S)) = h(S) \cup h(A - S) = f(S) \cup (B - f(S)) = B$. Thus h is a bijection from A onto B.

The next application leads to a proof of the Cantor–Bendixon theorem. For this we need a few concepts of topology.

Definition. If A is a subset of a topological space (X, τ), the derived set of A is the set of all accumulation points of A. This set is denoted by A'

A subset A of a topological space is said to be perfect if it equals its derived set (i.e. $A' = A$).

In a topological space a subset A is said to be scattered if the only subset B of A for which $B' \supseteq B$ is the empty set.

Theorem. (Cantor–Bendixon). *Every closed subset of a topological space (X, τ) is the disjoint union of a perfect set and a scattered set.*

Proof. Let A be a closed subset of X Define $B =$

$\cup\{S \in P(X) : A \cap S' \geq S\}$ and $C = A - B$.

The map $S \to A \cap S'$ on $P(X)$ the power of set of X is an isotone function on the complete lattice $P(X)$ into itself. So by Tarski's theorem B is a fixed point for this map. Thus $B = A \cap B'$ and $B' \geq B$. Since A is closed and $B \leq A$, we have $B' \leq A' \leq A$, it follows that $B = B'$ Hence B is perfect.

We now show that C is scattered. Suppose $S \subset C$ and $S' \supseteq S$. So $A \cap S' \supseteq S$ as $A \supseteq S$. By definition of $B, B \supseteq S$. Thus S is a subset of both B and $C = A - B$. This will be possible only if $S = \Phi$. This shows that C is scattered.

Example. For the closed subset $A = [-1, 0] \cup N \text{of} R$, the perfect part of A is $[-1, 0]$ while its scattered part is N

Concluding Remarks

Tarski's theorem is simple to state and admits an elegant proof. Further it has great applicability. The existence of certain formal languages with specific grammars can be established by an appeal to Tarski's theorem. This apart, it can be used to prove existence theorems for differential equations. Those who are introduced to lattice theory at an early stage, can get glimpses of its connections to other branches of mathematics such as topology, set theory and differential equations through Tarski's fixed point theorem. In fact, as shown by Davis [1], Tarski's theorem characterises complete lattices. In recent years, inspired by Tarski's theorem, mathematicians working in the area of functional analysis are studying similar fixed point principles in partially ordered Banach spaces.

Suggested Reading

[1] Davis C Anne, A characterization of complete lattices, *Pacific J. Mathematics*, 5, 311-319, 1955.

[2] A Tarski, A lattice-theoretical fixed point theorem and its applications, *Pacific J. Mathematics*, 5, 285-309, 1995.

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