Chaotic Dynamics on the Real Line

2. Non-periodic and Chaotic Behaviour

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Part 1. Periodic Behaviour and the Bifurcation Diagram, *Resonance*, Vol. 5, No. 4, p.52, 2000.

Recall that in the first part of this article, we discussed and illustrated stable periodic orbits for the logistic map using the bifurcation diagram. We now turn our attention to explore what happens to orbits of typical initial points $\{x\}$ under an f_λ which has no stable periodic orbit. Remark that whether a given logistic map has a stable periodic orbit or not cannot be decided using a computer alone. (This is because, all inputs and outputs on a computer are of finite precision; they handle only 8-digit or 16-digit decimals. In some of the latest software packages, we can specify a higher degree of precision but then more memory and more computer time will be required. It is clear that results obtained using single (8-digit) precision and using double (16-digit) precision can vary very much even for the same value of λ and x). For example, if we are using only two-digit precision any orbit will be periodic with a period at most 99. In general any orbit calculated using a computer will be periodic with a period less than 10^k where k is the number of digits after the decimal point. Of course this is a huge number even for $k = 8$. If the prime period of a stable periodic orbit is small enough (< 100, say) it may be easily located. However if its period is very large, like 10^{20}, say, the computer will not be able to locate it. Of course you may say that such large periodic orbits are not likely to be seen in practice. Theoretically however, we have to allow such possibilities.

So how do we find what happens at those values of λ where f_λ has no stable periodic orbit. It turns out that we can still take some help from the computer output which will now be a long series of say, eight-digit numbers without any apparent pattern. To make some sense out of such a large mass of numbers, anyone with a statistical bent of mind would suggest forming a 'frequency distribution' – divide the unit interval into a large (say 100) number of subintervals and find out how many points of the orbit fall into each of these. A histogram can also be plotted directly using the computer. We can get some idea of the orbit behaviour by looking at the histogram. If there is a stable periodic orbit we will naturally see a uniform distribution on the set of points in the periodic orbit.

When there is no stable periodic orbit, only two types of orbit behaviour are possible.

Theoretical breakthroughs in the study of the logistic and similar unimodal maps on the unit interval were made during the years 1975-80. The book by P Collet and J-P Eckmann [1] already gives a detailed and rigorous account of these developments but seems to have reached only the specialists and not the general mathematical world. It was proved that when there is no stable periodic orbit, only two types of orbit behaviour are possible. We will illustrate these two possibilities for the logistic family using histograms drawn by a computer program. *Figure* 1 gives the histogram of one lakh iterates of the critical point $c = 0.5$ under f_λ when $\lambda = 3.5699456$ and *Figure* 2 a similar histogram when $\lambda =$

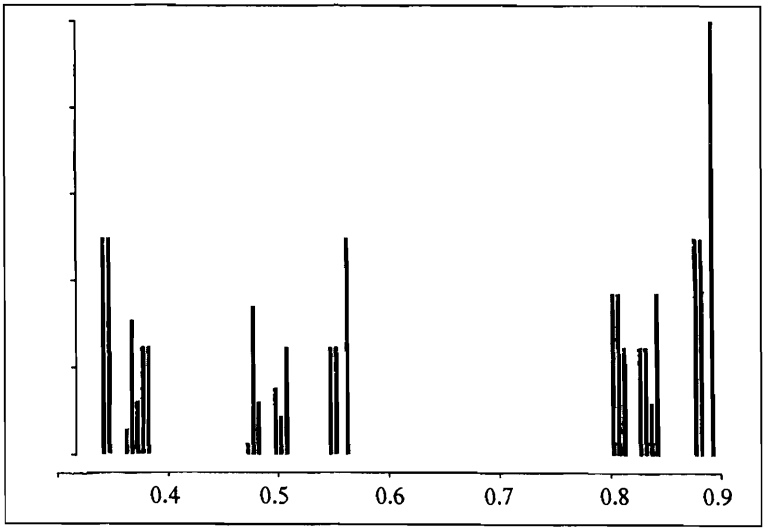

Figure 1. Histogram at λ= 3.5699456.

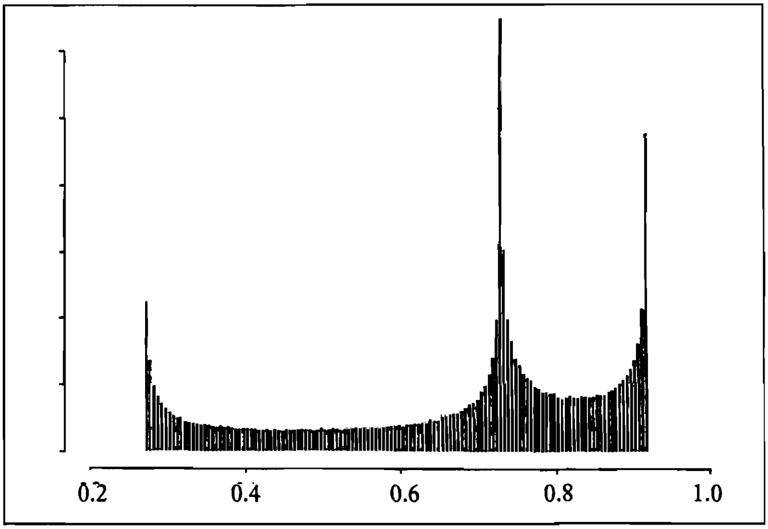

Figure 2. Histogram at λ= 3.6785735.

3.6785735. Anyone can easily see that there is a qualitative difference between these histograms. In the first case, the iterates seem to get confined to a rather strange set. The parameter value 3.5699456 is a good approximation (from the left) to the Feigenbaum point λ_∞ mentioned in part one and at this value there is in fact a stable periodic orbit of period 2048. At the Feigenbaum point itself, the attractor is a Cantor set and the histogram seems to give an indication of this. In the second case the histogram seems to approximate a smooth density function. The value of the parameter used now, 3.6785735, is an approximation to the 'first band-merging point', which is the value b of λ where the two bands seen in the bifurcation diagram merge into a single band (see *Figure* 3). Analytically, the first band-merging point is a solution of the equation in λ given by $f_\lambda^2(c) = 1/\lambda$; i.e., $\lambda^3(4-\lambda) = 16$ (see [2], p.98 for more details.)

The time has now come for us to agree on what we should mean when we say that a dynamical system behaves chaotically. It is of interest to note here how, historically, the nomenclature 'chaos' came into being. During the sixties and seventies, the literature on this

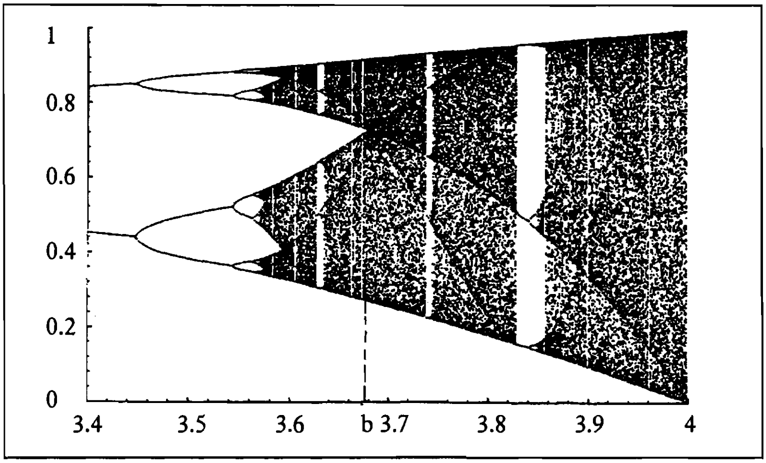

Figure 3. Enlarged bifurcation diagram.

topic was scattered in various journals – some in purely theoretical ones on dynamical systems (in two or more dimensions) by S Smale and others, some on the logistic map (or its sister the quadratic map) by Myrberg and by Metropolis, Stein and Stein [3], with applications in population biology by R M May [4] and some by E N Lorenz on the impossibility of long-term weather prediction. By a curious coincidence the paper 'Period three implies chaos' by Li and Yorke [5] in 1975 seems to have brought all these diverse investigations into a single focus by providing a technical and attractive name 'chaos' to the underlying phenomenon. In fact the definition of chaotic systems (to be introduced below) which is widely used nowadays is somewhat different from what Li and Yorke meant by chaos.

It is commonly acknowledged that 'sensitive dependence on initial conditions' is the hallmark of chaotic dynamical systems. This means that two orbits starting very close to each other eventually get separated as the system evolves. To put this mathematically, let (X, f) be a dynamical system (with its natural metric d). We say that the system has the property 'sensitive dependence on initial conditions' (SDIC, for short) if there exists a constant $\epsilon > 0$ such that for any x and any $\delta > 0$ there is a y which is δ-close to x and an integer k such that

$f^k(x)$ and $f^k(y)$ are at a distance greater than ϵ. (Actually we may be able to show this only for 'almost all' initial points x.) Thus if x is the true state of the system and y is very close to x (for instance, an approximation to x that is fed into the computer) and if the system has SDIC, then $f^k(x)$ and $f^k(y)$ could be very different and so taking the latter as the value of the orbit of x at time k would be erroneous. Thus prediction is impossible in systems with SDIC.

There is an index known as the 'Lyapounoff exponent' using which we can say whether a system has SDIC or not. This is theoretically elegant and practically useful. We can think of $d_0 = |x - y|$ as the initial error. The error in the observation at time n is then $d_n = |f^n(x) - f^n(y)|$ which is approximately $d_0(f^n)'(x)$, provided d_0 is small enough. Using the chain rule for differentiation and a few other mathematical results, the above error at time n, for large n, can be expressed approximately as $d_0\, exp(nI)$ where by definition,

$$I = I(x) = \lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(x)|$$

which is known as the Lyapounoff exponent at x. Clearly $I(x)$ is the rate of (exponential) growth of errors. For a large class of unimodal maps including the logistic family, it is known that (see [6], p.366) for 'almost all' initial points x, $I(x)$ is the same constant I and $I > 0$ if and only if f has an 'invariant density function'(see *Box* 1). If $I > 0$, then it can be shown that the system has SDIC and so is chaotic. But then which initial point x should we choose to find the Lyapounoff exponent? It turns out that we can take $x = f(c)$ because by virtue of another theorem, if

$$\liminf_{n\to\infty} \frac{1}{n} \log|(f^n)'(f(c))| > 0$$

then there is an (ergodic) invariant density function and the above remarks will show that the map f is chaotic.

Box 1. Invariant Density Function

A non-negative continuous function g on [0,1] with $\int_0^1 g(x)dx = 1$ is called a *density function*. This defines a *probability measure* P by the requirement that

$$P([a,b]) = \int_a^b g(x)dx$$

for all subintervals $[a,b]$ of $[0,1]$. The density function g (or the measure P) is called *invariant* for the dynamical system $([0,1], f)$ if

$$P(f^{-1}([a,b])) = P([a,b])$$

for all subintervals $[a,b]$ of $[0,1]$. When

$$f(x) = 4x(1-x)$$

it is clear that $f^{-1}([a,b]) = [a_1,b_1] \cup [a_2,b_2]$ where $[a_1,b_1] \subseteq [0,1/2]$ and $[a_2,b_2] \subseteq [1/2,1]$ can be calculated explicitly. Using these values and taking

$$g(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

it can be verified that g is an invariant density function for f.

We shall give two examples of maps from the logistic family which are chaotic. It was known as early as 1947 (to S M Ulam and John von Neumann) that the logistic map at $\lambda = 4$, viz.,

$$f_4(x) = 4x(1-x) \quad 0 \leq x \leq 1$$

has an invariant density function (see *Box* 1) given explicitly by

$$g(x) = \frac{1}{\pi\sqrt{x(1-x)}} \quad 0 \leq x \leq 1.$$

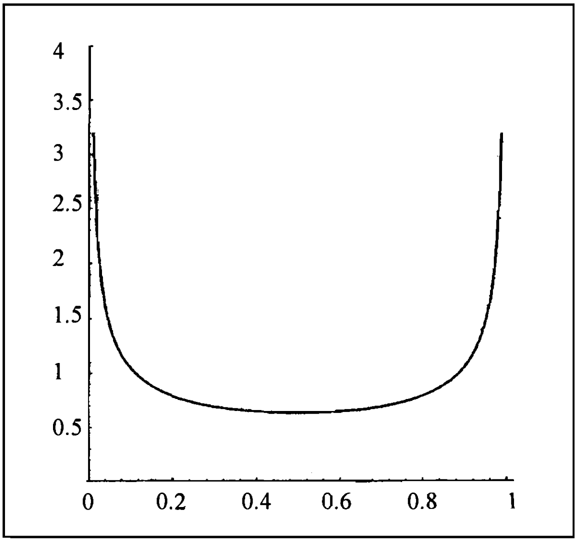

Figure 4. Graph of the arc sine law function.

This is known as the 'arc sine law' in the theory of random walks on the real line. The graph of this function is a U-shaped curve (see *Figure* 4). Hence f_4 is a chaotic map. One can also calculate the above limiting value as log 4. There are also other ways of proving that the map f_4 is chaotic. Another value of the parameter where there is an invariant density and the logistic map is chaotic is the first band-merging point b mentioned earlier. In this case the above limiting value can be found to be $\log|2-\lambda|$ and hence is positive since $\lambda > 3$. The histogram in *Figure* 2 clearly approximates a density function quite well. We leave it to the imagination of the advanced readers to guess where the other band-merging points are and why the logistic maps at those points will be chaotic.

Practically all works on popularisation of chaos theory (including [2], [7], [8] and [9]) assert that if a system does not have a stable periodic orbit then it is chaotic. But this is not true. The logistic map at the Feigenbaum point does not have a stable periodic orbit nor is it chaotic. The orbits of almost all initial points get into a certain Cantor set (the attractor) and once there, continue to move within it in a regular fashion; points

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close to each other stay close ever afterward. Of course this may not be seen in the histogram for the orbits of the critical point under the logistic map with parameter $\lambda = 3.5699456$, but can be proved theoretically (see [1] or [6]).

There are uncountably many values of the parameter where the logistic map has this kind of Cantor attractor. There are also uncountably many values of the parameter where the map has SDIC and so is chaotic. But the set of parameter values of the former type has Lebesgue measure zero whereas the set of parameter values of the latter type has positive Lebesgue measure (see [1] or [6] for these results). Indeed the latest result on these is that for 'almost all' values of the parameter, the logistic map has either a stable periodic orbit or an invariant density function ([10], pp. 326-327). Such results will surely kindle the interests of the mathematically oriented reader for further studies of the logistic and similar families on the unit interval.

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When research workers trained in different methodologies come together and contribute to a subject like chaos, it is only natural that there is a lot of confusion over terminology. Very often in higher dimensions, for a chaotic system the attractor turns out to be a 'fractal' and hence the name 'strange attractor' But in the case of the logistic family, we see that the logistic map at the Feigenbaum point has a Cantor set (which is a fractal) as the attractor, but the motion within the attractor is not chaotic. At the first band-merging parameter value mentioned earlier, even though the map is chaotic in the sense of having the SDIC property, the attractor is a union of intervals (see [1] or [6]) and so not so 'strange'. Examples of such systems have been given in two dimensions also. The word 'chaotic' refers to the dynamics of the system whereas the word 'strange' refers to the geometry of the attractor.

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Another paradoxical terminology that has become popular in this subject is 'deterministic chaos'. How can a system that evolves according to a deterministic model become chaotic and unpredictable? To see the apparent contradiction, let us look at the definitions again. If the system is modelled by (X, f) and if the state of the system is x to start with, then the model prescribes that the state of the system at time n is $f^n(x)$, a well-determined state. If we now ask how the system would behave over a distant 'future' – that is, *over a period of time in future*, how are we justified in looking for a deterministic answer unless there is a stable periodic orbit? The answer can only be a set of numbers like $\{f^m(x), f^{m+1}(x), \quad f^{m+k}(x)\}$ depending on the time period involved and the initial value x. If k is large, it makes sense to form a frequency distribution and give a statistical description. If there is a 'nice' invariant density then certain mathematical consequences may be worked out and verified experimentally. There is nothing much new in this approach – indeed the mathematical model is essentially the same as that for 'random' phenomena dealt with in probability theory where a density function (or more generally a probability measure) is assumed to be given on the 'sample space' (= phase space), and then the 'random variable' (the observable function f on the sample space) has a certain probability distribution. The main difference seems to be that in chaos theory the observable function is given on the phase space to start with and there may or may not be a suitable invariant density whereas in probability theory one has a suitable probability measure right at the start so that probabilities of events of the type: $f(x)$ belongs to a specific set, can be calculated.

All said and done, the practical utility of the chaos concept seems to arise from the fact that certain dynamical systems are parametrised in a natural way and as the parameter is increased (in general, changed) the system moves from a regular (periodic) behaviour to an irregular (chaotic) behaviour. We have seen that the logistic family takes the 'period-doubling route' to chaos; in higher dimensional systems, there are other 'routes' to chaos as well. The period-doubling route to chaos has been observed in practical experiments in physics, chemistry, electronics, etc. (see [11]). If a system is found to follow one of the routes to chaos, then it may be worthwhile to 'control' the parameter so that chaos is prevented from occurring. These are some of the reasons for the fact that chaos theory has become a centre of attraction in all the sciences.

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Suggested Reading

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