# Discrete-Time Systems

## 2. Where does the Vacillating Mathematician go?

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**In this part we analyze the discrete-time systems and suggest some intuitive design procedures. We survey the state-of-the-art digital control systems.**

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### Analysis of Discrete-Time Systems

### Example 1: The Vacillating Mathematician.

This problem was discussed at length by K B Athreya in [1]. However, in this article we present a control engineer's version. For the sake of clarity we restrict ourselves to the deterministic problem.

The sequence $\{y(k)\}_{k=0}^{\infty}$, $y(k)$ denoting the position at the $k^{\text{th}}$ change point, is $\frac{1}{2}, \frac{1}{4}, \frac{5}{8}, \frac{5}{16}, \cdots$

Let us split this sequence into two – the odd sequence and the even sequence, such that

$y(2k+1) = \frac{1}{2}(1 + y(2k))$ and $y(2k) = \frac{1}{2}y(2k-1)$.

Together, we obtain a second order difference equation

$$y(k+2) + \frac{1}{2}y(k+1) - \frac{1}{2}y(k) = \frac{1}{2} \tag{1}$$

in the advanced form. But in what follows, we shall employ the delayed form so that we have better physical interpretations. Accordingly, the initial conditions are $y(-1) = 0, y(0) = \frac{1}{2}$. We may now define the state variables $x_1(k)$ and $x_2(k)$ so that

$$A = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{2}$$

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Substituting and applying Z-transformation we get

$$y(k) = \frac{1}{2} + \frac{1}{6}(-1)^k - \frac{1}{6}(\frac{1}{2})^k \tag{3}$$

It is easy to see that the mathematician hops in the vicinity of $\frac{1}{2} \pm \frac{1}{6}$ asymptotically.

The following observations can be made from the equations.

[1] Let us look at the rhs. Apart from the initial conditions, the mathematician is driven by an impulse of strength $\frac{1}{2}$ at each time instant $k$.

[2] If he were not driven so, he would be hopping in the vicinity of his home! By linearity we can guess that he would as well be hopping in the vicinity of his office if each impulse is of unit strength.

[3] Thus, his very characteristic (zero-input response) of vacillation can be attributed to the presence of a pole at $z = -1$, i.e., on the unit circle in the complex z-plane. The other pole, of course, is a stable one at $z = \frac{1}{2}$. According to the standard definition, the mathematician is stable in the sense of Lyapunov (since his hops are confined to within $\pm\frac{1}{6}$ of an equilibrium point). See *Box* 1 for the definitions of stability.

Let us now verify if this mathematician can be *engineered* to reach either his home or his office at least asymptotically. By this, we mean to investigate the possibility of moving the unstable pole into the region of stability. For this, we need a very important concept called 'controllability' This is, in principle, equivalent to the condition of 'complete characterization' discussed in [3]. See *Box* 1 for a definition. Here we present the idea intuitively so that it would lead to the design of a suitable compensator.

If we apply Z-transform to the state equation and re-

**Box 1. Definitions**

**Stability in the sense of Lyapunov :** A system, or more precisely its zero input response, is said to be *marginally stable* or *stable in the sense of Lyapunov* if the response excited by every finite initial state is bounded. It is said to be *aymptotically stable* if the response excited by every finite initial state is bounded and approaches zero as $t \rightarrow \infty$. These definitions are applicable only to linear systems. For more general definitions that are applicable to both linear and nonlinear systems, see [2]. In terms of state space models, the eigenvalues of the system matrix A are required to lie within (or at the most on) the unit circle in the complex z-plane. Accordingly, this unit circle is also called the region of convergence.

**Controllability and Observability :** A system is said to be *controllable* if we can transfer any state to any other state in a finite time by applying an input. It is said to be *observable* if we can determine the initial state from the knowledge of the input and output over a finite time interval.

arrange the resulting form, we get

$$(zI - A)\vec{X}(z) - B\vec{U}(z) = z\vec{X}(0).$$

If we define a composite vector comprising the state vector and the input vector,

$$\vec{\zeta}(z) = \left[ \begin{array}{ccc} \vec{X}(z) & : & -\vec{U}(z) \end{array} \right]^T$$

then,

$$[zI - A : B]\vec{\zeta}(z) = z\vec{X}(0). \qquad (4)$$

This is a linear equation with $n + r$ unknowns and $n$ equations. (Notice that $\vec{X}(0)$ is available to us.) This has an unique solution $\vec{\zeta}\,(z)$ if and only if the composite matrix $[zI - A : B]$ has full rank $n$. It follows that each of $r$ unknowns exists and may be expressed as a linear combination of the other $n$ unknowns. This condition is what is referred to as *controllability*. Further, if we suppose that $z$ are *not* the eigenvalues of $A$, then, $[zI - A]$ itself has full rank. Accordingly, $\vec{U}$ may be found to be a linear combination of $\vec{X}$ Or, equivalently,

$$\vec{U} = -K\vec{X} + \vec{r}$$

for some $\vec{r}$. Notice that $K$ is a matrix. Thus, the actuating signal is a linear combination of the states! This is called the *state feedback control law.* We call $K$ the state feedback gain. To solve for $K$, we need to solve for $\vec{\zeta}(z)$ for each of the desired pole locations.

It is easy to verify (and hence left to the reader) that the mathematician is controllable. Now, let us move the pole on the unit circle to $z = -\frac{1}{2}$. Upon solving $\vec{\zeta}(z)$ for $z = \pm\frac{1}{2}$, we get $K = \begin{bmatrix} \frac{1}{2} & -1 \end{bmatrix}$ and hence

$$\begin{aligned} u(k) &= r - \frac{1}{2}x_1(k) + x_2\,(k) \\ &= r - \frac{1}{2}y(k-1) + y(k-2). \end{aligned} \qquad (5)$$

Here $r$ denotes *reference*, i.e., it refers to either the mathematician's home or his office in some sense. The state feedback control law suggests our mathematician to *observe* the past two hops and decide the current change point. If he wishes to reach home, the sequence is

$y(k) = \frac{1}{2}\left[(\frac{1}{2})^k - (\frac{-1}{2})^k\right]$.

The sequence converges to 0. We leave it to the reader to find an appropriate reference $r$ so that the sequence converges to 1.

A final comment is that we may place the poles anywhere within the unit circle so that the sequence always converges. For a different set of pole locations, we get a different gain matrix $K$

## Conclusions

The rapid development of digital technology, integrated circuits, and sophisticated high-density device fabrication techniques has radically changed the boundaries of

.practical control system design options. It is now routinely feasible to employ very complicated high order digital controllers and to carry out the extensive calculations required for their design. Digital controllers can also be reprogrammed for different tasks.

The area of digital control systems (or computer-controlled systems or sampled data control systems), is the technological area which aims to develop controller design methods based on digital computers, microcontrollers or more recently the DSPs. For more details on DSPs refer to [4] and [5].The results reported thus far are significant from both the theoretical and practical points of view. From the theoretical point of view, these results are presented in great depth, covering a wide variety of modern digital control problems such as optimal and stochastic control, adaptive control, state observers, Kalman filters and system identification. From the practical point of view, these results have been successfully implemented in numerous practical systems and processes. such as control of position, velocity, voltage, temperature, pressure, fluid level, electrical energy plants, industrial plants producing paper, cement etc., nuclear and chemical reactors, ground, sea and air transportation systems, robots, space applications, farming, biotechnology and even medicine.

The disciplines of continuous-time and discrete-time signals and systems have become increasingly entwined. Without any doubt, it is advantageous to process continuous-time signals by sampling them. The computer control system for a modern high-performance aircraft, for example, digitizes sensor outputs such as the vehicle's speed, in order to produce a sequence of sampled measurements which are then processed. Yet another example could be BSE (Bombay Stock Exchange) index.

Despite these advantages, digital control systems do pose several problems such as appropriate sampling rates (i.e., aliasing), scaling and round-off errors. Nevertheless, the advantages far outweigh the disadvantages and digital systems continue to pervade every walk of our lives. Digital televisions, digital versatile discs are but a few to mention.

## Suggested Reading


[1] A Ramakalyan and J R Vengateswaran, Systems and Control Engineering, *Resonance*, Vol. 4, Nos. 1, 3 & 5, 1999.

[2] K B Athreya, The Vacillating Mathematician Problem, *Resonance*, Vol. 2, No. 1, 1997.

[3] V Rajaraman, Digital Signal Processors, *Resonance*, Vol. 4, No. 6, 1999.

[4] Farzad Nekoogar & Gene Moriarty, *Digital Control Using Digital Signal Processing*, Prentice Hall, 1999.

[5] C T Chen, "*Analysis and Design of Control Systems: Analog and Digital*", Holt, Rinehart and Winston, N.Y., 1993 (Also available from Oxford Univ. Press priced in rupees.)

[6] C T Chen, *Linear System Theory and Design*, New York: Holt, Rinehart and Winston, 1984. (Also available from Oxford Univ. Press priced in rupees.)



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Both Szilard and Einstein were theoreticians; but both had a complementary side to their interests, as though a miniature-painter were to take up carpentry as a hobby. Thus Szilard had a practical inventive flair that tied in with Einstein's long experience in the Berne Patent Office. One result was a series of joint patents, lodged in Britain and the United States as well as in Germany, for what was then a revolutionary form of heat exchange refrigerator. Some re-collections claim that Elsa expected Einstein to make a fortune from the patents; others, more plausibly, claim that the hopes were Szilard's. Little came of the scheme although the Einstein–Szilard heat-pump, which provides its essential mechanism, has become a feature of many post-war nuclear power stations.

From: *Einstein – The Life and Times*
by R W Clark