Chaotic Dynamics on the Real Line

1. Periodic Behaviour and the Bifurcation Diagram

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Introduction

Chaos is the order of the day. The modern concept of chaos is now finding applications in several branches of science. Readers of *Resonance* have already been exposed to some of the basic ideas of chaos through the articles of Ramasamy and Iyer [1] and Krishan, Manu and Ramaswamy [2]. It has been termed "one of the major discoveries of our times and a new way of Scientific Thinking" [1]. The early book [3] by James Gleick brought the basic ideas close to the layman. Ten years ago, controversies raged among mathematicians and chaos theorists about the importance of the idea of chaos (see various articles in the *Mathematical Intelligencer*, 1989). Mathematicians in general tend to hold the view that there are hardly any rigorous results in chaos theory as the conclusions are largely based on computer simulations – these are essentially periodic in nature because of the finite precision involved in calculations by computer. However such computer simulations have actually drawn the attention of mathematicians to unexplored areas begging for explanations of 'bizarre' behaviour of dynamical systems – especially in one and two dimensions. Indeed there has been vigorous mathematical activity among the specialists (which continues to this day) especially on discrete one-dimensional systems; by now these are fairly well understood [4]. More recently we also have popular books written by expert mathematicians like Stewart [5].

'Chaos theory' is about the behaviour of dynamical systems in the long run. We start with a non-empty set X of points x called the 'phase space'. Very often X is a subset of a finite dimensional Euclidean space. The elements of the phase space represent states of the system under study. The system evolves over a period of time according to certain laws, assumed to be given either by the solution of a system of differential equations or by a continuous function on the phase space (with its natural metric). The example of a simple pendulum belongs to the former category while that of the logistic model for population growth is a typical example of the latter situation (see [2]). In general, solving the system of differential equations is pretty difficult and so numerical approximations to the solutions are used to study the system. Even for the case of the explicitly given logistic map, numerical studies were resorted to in the early stages, since the properties of the dynamical system generated were not easy to guess theoretically, let alone prove them. Thus the computer has been an essential tool in the development of chaos theory.

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In this two-part article we shall exhibit a number of interesting properties of the dynamical system defined by the logistic map on the unit interval of the real line. This has been acclaimed to be the simplest system with which one can start learning about chaos. In the first part we give details about convergence of typical orbits to a 'stable' periodic orbit which is also captured beautifully by the bifurcation diagram, drawn using a computer program. In the second part, we shall turn to non-periodic behaviour and indicate how computer studies are of only limited help. We shall also see that contrary to what is generally claimed (including [1], [2], [6] and [7]), absence of a 'stable' periodic orbit does not imply chaotic behaviour, at least in the case of one dimensional systems. (Of course we should agree on what we understand by chaos. In a series of recent articles on 'Clarifying Chaos', R Brown and L O Chua present nine different definitions of chaos and using examples and counterexamples, conclude that "it appears that for any given definition of chaos, there may always be some 'clearly' chaotic systems which do not fall under that definition, thus making chaos a cousin to Gödel's undecidability" These articles have appeared in the *International Journal of Bifurcation and Chaos* between the years 1996 and 1999.)

Generalities

The basic 'phase space' for us is the unit interval $X = [0,1]$. Any continuous function f on X into X defines a (discrete) dynamical system. If x is the state of the system to start with, then $f(x)$ is its state after one unit of time; its state after two units of time would be $f(f(x)) = f^{(2)}(x)$, the second iterate of x. In general the state of the system after n units of time is denoted by $f^{(n)}(x) = f(f^{(n-1)}(x))$ the n^{th} iterate of x under f. For brevity we write only $f^n(x)$ for $f^{(n)}(x)$ for any n, x with $f^0(x) = x$ and $f^1(x) = f(x)$. To study the long term behaviour of the system we therefore need to study the *orbit* $\{x, f(x), f^2(x), \quad , f^n(x), \quad \cdot\}$ of x under f.

The simplest behaviour is when the orbit repeats itself after a certain stage – leading to periodic behaviour. We make the following definition:

Definition : A point $p \in X$ is called a *periodic point* of prime period k if $p, f(p), f^2(p), \quad , f^{(k-1)}(p)$ are all distinct and $f^k(p) = p$. In this case note that each $f^i(p)$ is also a periodic point of prime period k. When p is a periodic point, its orbit is called a *periodic orbit* and is denoted by the finite set $\{p, f(p), \quad , f^{(k-1)}(p)\}$. If $f^m(p) = p$ for some m, we say m is a period of p. If $m = 1, p$ is called a fixed point. If p is not a periodic point but $f^m(p)$ is a periodic point for some $m > 1, p$ is called an eventually periodic point.

Definition The periodic point p is said to be *stable* (or *attracting*) if there is a neighbourhood U of p such that for all $x \in U$, the orbit of x has the property that $f^{kn}(x)$ converges to p as $n \to \infty$. In this case by continuity, $f^{(kn+i)}(x)$ converges to $f^i(p)$ and so we can say that the orbit of x converges to the periodic orbit $\{p, f(p), \quad f^{(k-1)}(p)\}$.

If p is a stable periodic point, then in fact, the orbits of 'most' points in X will converge to the periodic orbit containing p (see [4]). In this case the periodic orbit may be called the *attractor* of the dynamical system. In general the attractor of a dynamical system is the subset of the phase space to which the orbits of 'most' of the initial points will converge. (Here 'most' can be interpreted in a topological or a measure theoretic sense.)

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By a simple application of the mean-value theorem of elementary calculus, one can prove that the periodic point p is stable if $|(f^k)'(p)| < 1$. (See e.g., [7].) Since $(f^k)'(p) = \prod_{i=0}^{k-1} f'(f^i(p))$ by the chain rule for differentiation, $(f^k)'(p) = (f^k)'(f^i(p))$ for all i and so if p is stable, so are $f^i(p)$ for all i – as they should be. We can as well say that the periodic orbit is stable if $|(f^k)'(p)| < 1$. If $|(f^k)'(p)| > 1$, then it can be shown that there is a neighbourhood U of p such that the orbit of any $x \in U$ except that of p itself moves out of U sooner or later and there is no convergence to the periodic orbit. In this case p is called an *unstable* (or *repelling*) periodic point. The case when the derivative is equal to 1 is a little complicated and we do not need to enter into a discussion on that here.

The Logistic Family

Let us now take up the study of stable periodic orbits for the logistic family. This family is defined as follows :

$$f_\lambda(x) = \lambda x(1-x) \quad 0 \leq x \leq 1$$

where λ is a parameter allowed to vary in the interval $[0,4]$. For fixed λ, f_λ maps $[0,1]$ strictly into $[0,1]$ except when $\lambda = 4$ in which case f_λ is onto. Each f_λ is an inverted parabola with a maximum $(= \lambda/4)$ at $x = 0.5$ f_λ is strictly increasing in $[0, 0.5]$ and strictly decreasing in $[0.5, 1]$. (See *Figure* 1.) The point 0.5 is called the critical point and is usually denoted by c. For a given λ, f_λ is referred to as a logistic map.

The case when $0 < \lambda \leq 1$ is easily dealt with, since for each $x \in [0, 1]$, $f_\lambda^n(x)$ converges to 0. So from now on we will consider only the case $1 \leq \lambda \leq 4$.

Analysis of what happens when $1 < \lambda \leq 3$ is easy enough. The equation $f_\lambda(x) = x$ has two solutions $x = 0$ and $x = (\lambda - 1)/\lambda$ when $\lambda > 1$. These are the fixed points of f_λ. Moreover since $f'_\lambda(x) = \lambda - 2\lambda x$, we get $f'_\lambda(0) = \lambda > 1$ and $f'_\lambda(p_\lambda) = 2 - \lambda$ where $p_\lambda = (\lambda - 1)/\lambda$ is the other fixed point. Thus, when $\lambda > 1$, 0 is an unstable fixed point and p_λ is stable as long as $|2 - \lambda| < 1$ or $\lambda < 3$. Hence, we know that for all x in a neighbourhood of p_λ, the orbit of x converges to p_λ. Indeed by graphical and analytical considerations, one can prove that for all $x \in (0, 1)$, the orbit of x converges to p_λ.

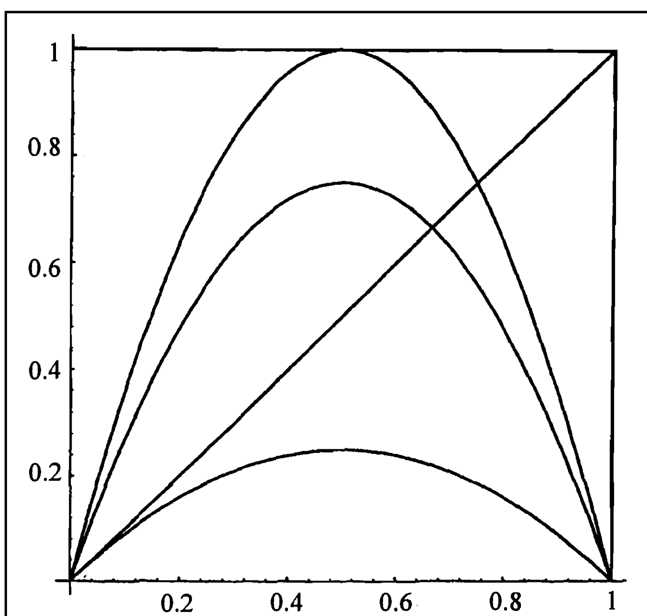

Figure 1. Logistic maps at $\lambda = 1, 3, 4$.

At this point, it may be worthwhile to use a computer to study what happens when λ is in the interval $[1, 3]$ as well as when λ is increased above 3. It is easy to write a program in *Basic* for computing the orbit of an initial value x under the logistic map f_λ. You can check that for $\lambda = 1.6, x = 0.05$, (or indeed any x that you can enter into the computer), the orbit converges to 0.375. For $\lambda = 2.3$ (say), the orbit of any x converges to 0.5652174. At $\lambda = 3$ itself, you will see that the orbit converges very slowly to .666667. At $\lambda = 3.1, x = 0.4$, the orbit settles down to the period two orbit $(.7645665, .5580142)$. At $\lambda = 3.4, x = 0.7$ also the orbit settles down to a period two orbit, but now to $(.8421544, .4519633)$. At $\lambda = 3.45$, however, the limit set turns out to be a period four orbit.

Thus it appears that convergence to a fixed point for $\lambda \leq 3$ gives way to convergence to a period two orbit for $\lambda > 3$ but around 3.45, there is convergence to a period four orbit. There facts are easy to understand if we look at the graphs of f_λ and f_λ^2 for values of λ around 3. *Figure* 2 gives the graphs of f_λ and f_λ^2 for $\lambda = 2.8, 3$ and 3.2, respectively. The fixed points 0 and p_λ are also fixed points for f_λ^2 and for $\lambda < 3$, there are no other fixed points for f_λ^2. However the graph of f_λ^2 becomes tangential to the diagonal line at $\lambda = 3$ implying that p_λ is a triple root of the equation $f_\lambda^2(x) = x$ and as λ is raised above 3, two real roots $p_{1\lambda}, p_{2\lambda}$ say, are 'born' to $f_\lambda^2(x) = x$ (which is a fourth degree equation). Also one can see graphically, or calculate the roots explicitly and see that the derivative $f_\lambda'(p_\lambda)$ (which $= -1$ at $\lambda = 3$) decreases below -1 as λ is increased so that the fixed

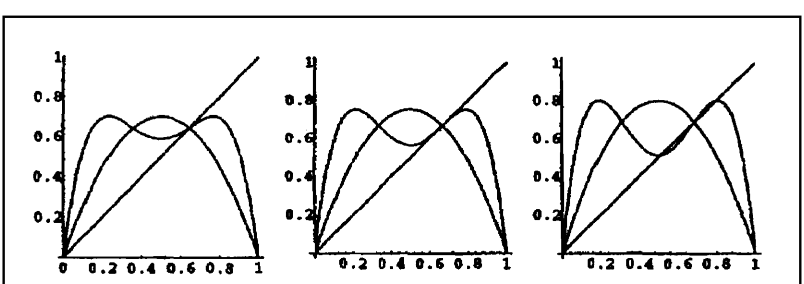

Figure 2. Graphs of f_λ and f^2_λ at $\lambda = 2.8, 3, 3.2$.

point p_λ becomes unstable and the two new roots $p_{1\lambda}$ and $p_{2\lambda}$ are such that $f_\lambda(p_{1\lambda}) = p_{2\lambda}, f_\lambda(p_{2\lambda}) = p_{1\lambda}$ and $|(f_\lambda^2)'(p_{i\lambda})| < 1$ for an interval of values of λ. Indeed this derivative is $+1$ at $\lambda = 3$, decreases as λ increases, attains the value 0 at $\lambda = 3.236$ and becomes -1 at $\lambda = 1 + \sqrt{6}$. If λ is increased above $1 + \sqrt{6}$, the same story unfolds between f^2 and f^4 as can be studied using graphs drawn by the computer (see [7]) and for an interval of values of λ, the period four orbit is stable and the orbit of any point in $(0, 1)$ (except those which eventually end up at the fixed point p_λ or the period two points $p_{1\lambda}$ and $p_{2\lambda}$) converges to the period four orbit.

As λ is increased further, the stable period four orbit becomes unstable and gives way to a stable period eight orbit and so on. This continues until stable periodic orbits of period 2^k for every k have appeared. This is the well-known period-doubling phenomenòn well explained in all books and articles including [1], [2], [6] and [7].

If we denote by λ_k, the value of λ at which a stable period k orbit becomes unstable and a stable period $2k$ orbit is born, then λ_k increases with k and so converges to a value λ_∞ but $\lambda_{k+1} - \lambda_k$ decreases very rapidly to zero. Correct to eight decimals, $\lambda_{14} = 3.56994567 = \lambda_{15}$ (see [8]). Another very interesting discovery about these $\lambda's$ was made by M J Feigenbaum (see [1], [7] or [8]). He found that

$$\lim_{k \to \infty} \frac{\lambda_k - \lambda_{k-1}}{\lambda_{k+1} - \lambda_k}$$

exists and is a constant (approximately 4.669). He actually discovered that this period doubling phenomenon and the value of the above limit are universal properties for a class of unimodal maps on the unit interval. A mathematical theory has also been developed explaining this using the renormalisation group ideas in physics. Nowadays λ_∞ is referred to as the Feigenbaum point.

Upto now, we have seen that for a given parameter value $\lambda < \lambda_\infty$ there is only one stable periodic orbit of period 2^m for some m. What happens when $\lambda > \lambda_\infty$? Early workers like R M May [9] called the interval $[\lambda_\infty, 4]$ the 'chaotic region'. What happens in this region can be picturesquely described by the bifurcation diagram. Before introducing this, let us note a few theoretical results.

In 1978, D Singer (see [4], p.155) proved an important theorem which helped development of the analysis of unimodal maps on the interval. He proved that for a large class of unimodal maps on an interval including the logistic map f_λ, there is at most one stable periodic orbit and if there is one such, then the orbit of the critical point will converge to it. Moreover, early workers like May [9] guessed that the set of parameter values λ where f_λ has a stable periodic orbit must be a 'dense' set in the interval $[1, 4]$. This has been proved to be true recently [10]. These facts suggest that if we plot the set of limit points of the orbit of c on the vertical axis against the parameter λ on the horizontal axis, we should get a good picture of what happens in the chaotic region. The resulting picture is known as the bifurcation diagram and it is the central theme of several mathematical investigations during the late seventies and eighties. This is drawn using a computer as follows. (There are programs in *Basic* in [8] and in *Mathematica* in [7] which can be used to draw this.) We take λ on the horizontal axis and divide the interval at (say) 400 equally spaced points. At each such λ, we plot $\{f_\lambda^n(c)\}$ on the vertical axis for $n = 300$ to 500 say. We start with $n = 300$ so that initial variations are removed and the orbit settles down to a periodic behaviour if λ is such that there is a stable periodic orbit for f_λ. If we take too many values of n, there will be too many black dots on the vertical axis (when there is no stable periodic orbit) since each point has some thickness and the size of the computer screen is limited.

Suggested Reading

[1] B Ramasamy and T S K V Iyer, Chaos Modelling with Computers, *Resonance*, Vol.1, No.5, 1996.

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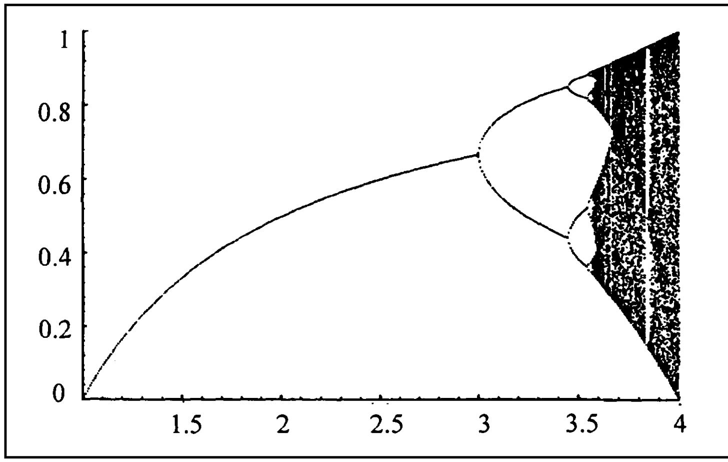

Figure 3. Bifurcation diagram.

Figure 3 gives the bifurcation diagram (for $1 \leq \lambda \leq 4$) drawn using the *Mathematica* program in [7]. We see the period-doubling phenomenon in the initial part of the diagram. *Figure* 4 gives an enlargment of the diagram for $3.4 \leq \lambda \leq 4$. In this we can see several (vertical) white strips. Each of these is usually called a 'periodic window' The widest of them is the 'period 3 window' which starts with a stable period 3 orbit. We can also see windows with period 5 (for $\lambda = 3.74$), period 6 (for $\lambda = 3.63$) and a period 4 window near 3.96 which however is much smaller. Each window starts with a stable periodic orbit of period k, say, undergoes a period doubling phenomenon and then enters a 'chaotic

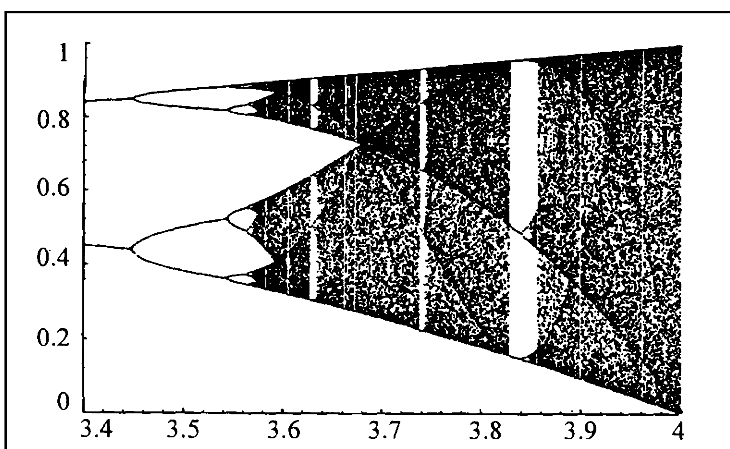

Figure 4. Enlarged bifurcation diagram.

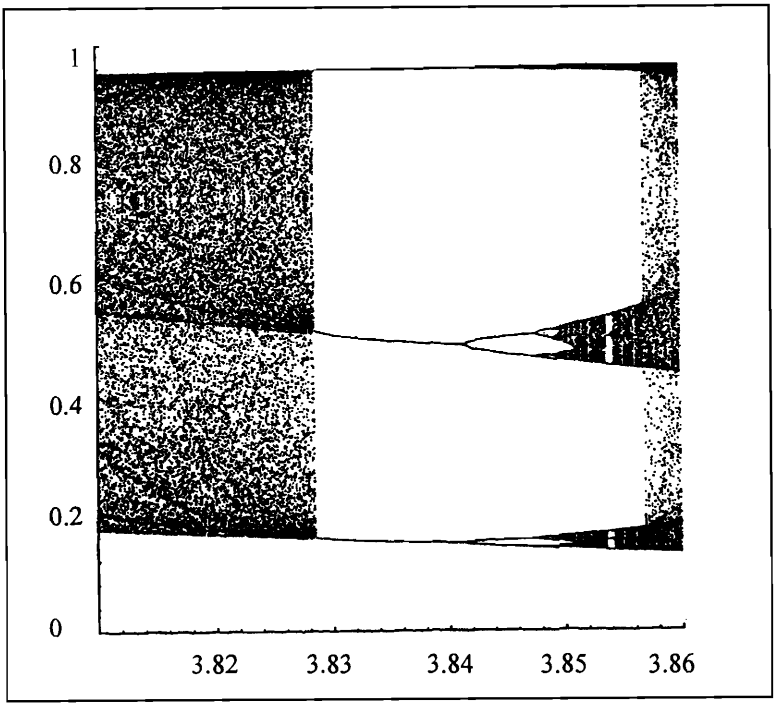

Figure 5. Period three window.

region'. This is best seen by enlarging the period 3 window; see *Figure* 5 which enlarges the bifurcation diagram for $3.80 \leq \lambda \leq 3.86$.

Many of the features that we see in the bifurcation diagram have been well understood – the period-doubling phenomenon in the initial part, the existence of stable periodic orbits of a given period for an interval of values of the parameter, the automatic formation of various curves within the diagram and most interesting of all, copies of the full diagram on the interval $[1, 4]$ appearing twice in the diagram for the interval $[3, 3.67]$ and thrice in the period 3 window, etc. Strang [11, p.510] describes drawing the diagram as the most impressive experiment that his book can propose.

What happens at those values of λ for which f_λ has no stable periodic orbit? It turns out that two distinct types of orbit behaviour are possible, only one of which may be called 'chaotic' We shall discuss this and some other related results in the next part of this article.

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