Knots and Surfaces

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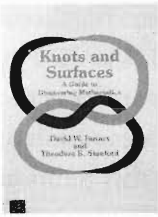

Knots and Surfaces
D W Farmer and T B Stanford
Universities Press, 1996
Price:Rs.85.

It would not be an exaggeration to say that this is the first time that a popular introduction to that fascinating branch of mathematics known as topology, has been published in our country. It is one of the volumes in a series of titles being brought out by the Universities Press and it is this reviewer's earnest hope that their endeavour generates enough enthusiasm for them to continue with it, as well as induce other publishers to pursue this direction.

The book starts out with an informal introduction to elementary graph theory. The Euler formula for planar connected graphs is recorded, though not proved. Of course, the proof is not difficult, and the authors could as well have included it, though they do mention in passing (Note 1.12e) that it can be done by induction on the number of vertices and edges. Euler's famous solution to the Königsberg bridge problem (which also resolves the school puzzles about tracing various networks without lifting the pencil or retracing) is also detailed. One concrete application of the Euler formula, to wit that six colours are sufficient to colour any planar map so that two countries sharing a border do not have the same colour, is proved in detail at the end of the first chapter. (It is worth remarking here that a little extra effort shows that five are sufficient, though the famous four-colour conjecture, i.e. that four colours will suffice, remained open for nearly a century before it was finally settled by Haken and Apple in 1974 after an extensive use of the computer).

Chapter 2 devotes itself to topological properties of surfaces, the important notion of simple-connectivity in particular, and how this important attribute is enough to distinguish between the torus and the sphere. Related to simple-connectivity is the Euler characteristic, which is the number $v-e+f$, where v is the number of vertices, e the number of edges and f the number of faces in a triangulation or cell-decomposition of the surface. *A priori* it is not even clear that this number will not depend on a particular choice of a cell-decomposition. However, and this is one of the deep consequences of homology theory (invented by Poincaré and Riemann), that it is in fact a *topological* invariant. Indeed, there is more! This integer is a *complete* invariant of all oriented (=two sided), connected boundaryless compact surfaces. In other words, if somebody gave you two very complicated looking surfaces of this kind, and asked you whether you could continuously bend, twist, stretch, shrink one of them into the other, without tearing anywhere, or using any glue, (i.e. to decide whether these two surfaces are *topologically* *equivalent*) you do not have to undergo the torture of actually trying. In fact, if you try and succeed, you have a "yes" answer, but if you do not succeed, that just means (i) it may be impossible, or (ii) you didn't try enough. Mathematics (to some, the reason for its beauty, to others, the reason for its unpalatibility) does not accept "may be" for an answer. So all you need to do with these two surfaces is to make out (with a pencil) a network or graph on it, so that this network subdivides your surfaces into 2-cells. A 2-cell is something equivalent to a 2-dimensional disc. For example a square, or a triangle, or anything which does not have holes in it. (That such a cell-decomposition always exists is again a deep fact!) Then just compute the Euler characteristic $v-e+f$, where f is the number of faces or 2-cells, e is the number of edges and v the number of vertices of the network you drew. If this Euler characteristic is different for the two surfaces, they are not equivalent, and if they are the same, they are equivalent. Unfortunately, the Euler characteristic cannot tell you, ahead of time, whether a boundaryless surface is two sided or not (for example, the Euler characteristic of a Klein bottle and a torus are both zero), but again, if you know that it is not two-sided, then again the Euler characteristic is a complete invariant for one-sided compact, connected surfaces. Also, a two sided surface is never topologically equivalent to a one-sided surface, and you can easily determine if a surface is two sided or one-sided by starting to paint it without lifing your brush, and seeing if the whole thing gets coloured or not. Since the surface is compact, you will finish this in finite time. Thus you have a complete solution to the problem of topological equivalence of compact, connected boundaryless surfaces.

The Euler formula for planar graphs leads to a classification of regular polyhedra into the 'Five Platonic Solids', as is explained in Section 2.5. The chapter closes with a discussion of cell-complexes, and also how it is possible to obtain any compact, connected surface by gluing the edges of a polygon in a prescribed manner. (Thus, any compact, connected surface has a cell-decomposition in which there is only one 2-cell, i.e. $f=1$!)

The last chapter launches knot theory, another old, but still very active, province of topology. The question is again, to find ways of distinguishing between knots. Of course, you are not allowed to cut the knot, and retie it. How many different knots are there, and are there invariants that can tell them apart? Amazingly, until a few years ago it was not even known whether the right and left-handed trefoil (see Figure 3_1 on p.84 of the book) were equivalent to each other. Most people believed they weren't, but no proof had been found. The most fundamental knot invariant, the Alexander polynomial, turns out to be the same for both. That they are inequivalent was one of the great successes of the Jones polynomials, invented by V F R Jones (of 1990 Fields Medal fame). Then the book goes on to discuss links and linking numbers, the mathematical basis for several conjurers' tricks that the reader must have doubtless encountered as a child.

The book's style is relaxed, and depends heavily on reader participation, as indeed, any really good mathematical exposition should. Much mathophobia would be dispelled if our school and college mathematics training did not turn the subject into feats of manipulative jugglery for the teacher, and a spectator sport for the students, who are then required to duplicate those stunts in an exam.

One fault of the book is that it sometimes forgets that it is addressing a high-school or college undergraduate audience, and 'talks down' to the reader. For example, Task No. 3.2.2: Make up some mathematical puns using 'un', 'not', 'knot' and 'unknot' or after Task 1.5.4: 'Euler is pronounced 'Oiler'. Say it aloud a few times. This will keep you from looking foolish later' are rather silly things to be telling an audience that has crossed primary school.

Also, the other fault with the book is that it does not point to some of the classics in topology, which are also quite accessible to an intelligent lay reader with some mathematical background. For example Lefschetz's *'Introduction to Topology'* and *'Topology'*, Chinn and Steenrod's *'First Concepts of Topology'*, Crowell and Fox's *'Introduction to Knot Theory'* and Seifert and Threlfall's *'Textbook of Topology'* are not mentioned at all, and I would urge any reader who is excited by the book under review, to get hold of the above references and at least dip into these famous books for a while. With these caveats, I would recommend it whole-heartedly to all high-school and first years' bachelor's students as a well-written and user-friendly first introduction to the subject.

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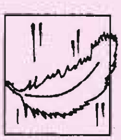

The belief in an external world independent of the perceiving subject is the basis of all natural science. Since, however, sense perception only gives information of this external world or of 'physical reality' indirectly, we can only grasp the latter by speculative means. It follows from this that our notions of physical reality can never be final. We must always be ready to change these notions – that is to say, the axiomatic structure of physics – in order to do justice to perceived facts in the most logically perfect way.

Albert Einstein