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Largest Two Entries in a Row and Column

Solution to a problem which appeared in 'Puzzling Rectangles Revisited', *Resonance*, Vol. 3, No. 11, pp. 87 – 88, 1998.

Let A be an $n \times n$ real matrix. Suppose x_i is the largest element in the ith row of A and y_i is the largest element in the ith column of A. Then clearly

$$\max_i x_i = \max_i y_i$$

since both sides equal the largest element in A. In particular, if we are told that $x_1 = x_2 = \ldots = x_n = x$ and $y_1 = y_2 = \ldots = y_n = y$, then we would readily conclude that $x = y$.

Now suppose u_i is the sum of the elements in the ith row of A and v_i is the sum of the elements in the ith column of A. Then

$$\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i$$

since both sides equal the sum of all the elements in A. Again we have a similar situation as before. Thus if we are given that $u_1 = u_2 = \ldots = u_n = u$ and $v_1 = v_2 = \ldots = v_n = v$, then we conclude that $u = v$.

Are there other functions which exhibit a similar behaviour? Curiously, if we consider the sum of the largest two entries in a row or column then we have a similar phenomenon. This is the essence of the following problem.

Problem 1: Let A be an $n \times n$ matrix with $n \geq 2$. Suppose that the sum of the largest two entries in every row of A is α and that the sum of the largest two entries in every column of A is β. Show that $\alpha = \beta$.

We will give several solutions. We start with a rather straightforward approach. For convenience, let R and C, respectively, denote the assumed row and column property of A. We assume, without loss of generality, that $\alpha \leq \beta$ and try to show that $\alpha < \beta$ cannot occur. Let if possible, $\alpha < \beta$. By property C, each column has at least one entry greater than or equal to $\beta/2$. On the other hand, by property R, since $\alpha < \beta$, in each row there is at most one entry greater than or equal to $\beta/2$. Since properties R and C are preserved under row and column permutations, we may assume that the main diagonal entries of A are all greater than or equal to $\beta/2$ and furthermore, they are in increasing order along the diagonal. Now by property C,

$$a_{j1} = \beta - a_{11}$$

for some $j > 1$. Then

$$a_{j1} + a_{jj} = \beta - a_{11} + a_{jj} \geq \beta > \alpha.$$

Thus the largest two entries in the jth row must add upto a number greater than α, which contradicts property R. Thus $\alpha < \beta$ cannot hold and the proof is complete.

The next solution we present makes use of some elementary concepts from graph theory. In each row of A circle the two largest elements. (In case of a tie the choice of an element is arbitrary.) Consider the bipartite graph G obtained as follows. The set of vertices is $\{1, 2, \ldots, n\} \cup \{1', 2', \ldots, n'\}$ and there is an edge from i to j' if and only if a_{ij} is circled. Thus we have a graph of $2n$ vertices and $2n$ edges. A very elementary argument shows that a graph of $k > 2$ vertices and k edges must contain a cycle: if the graph is connected then it cannot be a tree, since in a tree of k vertices there are $k - 1$ edges. Thus it contains a cycle. If the graph is disconnected, it must have a connected component with at least as many edges as vertices and again we have the same conclusion. Thus the graph G that we constructed has a cycle. Denote the set of entries of A constituting the cycle by C. Since G is bipartite, C has an even number, say $2m$, of elements. Let s be the sum of the elements of C. Then $s = m\alpha$, adding up the elements in each row first. However if we add the elements first column wise, then we conclude $s \leq m\beta$. Hence $\alpha \leq \beta$. A similar argument shows that $\beta \leq \alpha$ and the result is proved.

A slight modification of the above argument can be employed to prove a stronger result, which we state as our next problem.

Problem 2: Let A be an $n \times n$ matrix, $n > 2$. Let α_i and β_i be the sum of the largest two elements in the ith row and in the ith column respectively, $i = 1, 2, \ldots, n$. Show that there exist j, k such that $\alpha_j \leq \beta_k$.

To solve the problem we proceed with the previous argument. Furthermore we assume that the entries of A constituting the cycle C are located in rows $i_1, \ldots, i_m$ and columns $j_1, \ldots, j_m$. Then we get

$$\alpha_{i_1} + \ldots + \alpha_{i_m} = s \leq \beta_{j_1} + \ldots + \beta_{j_m}$$

for all j, k. It follows that $\alpha_j \leq \beta_k$ for some j, k.

Our next solution of Problem 1 has the same basic idea as the first solution but it executes it more efficiently. Let β_i be the largest entry in the ith row, $i = 1, 2, \ldots, n$. Then observe that $\alpha_i \geq \alpha/2$. We assume, by permuting the rows if necessary, that

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n.$$

There exists $j > 1$ such that

$$a_{1j} = \alpha - \alpha_1.$$

The sum of any two elements of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n, a_{1j}\}$ is at least α. Because two of the elements must be in the same column (pigeon-hole principle!), we must have $\beta \geq \alpha$. Using the same argument with the transpose of A we get $\alpha \geq \beta$ and the argument is complete.

Our final solution is the least intuitive and therefore has an element of surprise. Visit entries of A successively on a 'largest-entries-first' basis. Let $x \geq y$ be the values of the entries seen in visits number n and $n + 1$. There exists a row not visited during visits $1, 2, \ldots, n-1$, and hence the sum of the two largest entries in that row is at most $x+y$. Also there exists a row visited twice during visits $1, 2, \ldots, n+1$, and the sum of the two largest entries of that row is at least $x+y$. It follows by the hypothesis on A that $\alpha = x + y$. Using a similar reasoning for columns we get $\beta = x+y$.

The idea behind the previous solution allows us to formulate more general versions of Problem 1. The rows and columns of a matrix are just two ways of partitioning the entries of a matrix. We might as well consider an arbitrary set partitioned in different ways. We illustrate this by an example, given as the next problem.

Problem 3: Suppose a cube with its sides parallel to the co-ordinate planes is divided into n^3 smaller cubes, and each smaller cube is assigned a real number. Suppose the sum of the largest two entries in any cross-section parallel to the xy-plane is α and the sum of the largest two entries in any cross-section parallel to the yz-plane is β. Show that $\alpha = \beta$.

Just as before, visit the entries of the cube successively on 'largest-entries-first' basis. Then both α and β equal the sum of the values of the nth and $(n+1)$-th largest entries in the cube.

How about taking the sum of the three largest entries instead of two? The following example shows that the result is no longer valid. Consider

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 3 & 0 & 2 & 2 \\ 0 & 3 & 2 & 2 \\ 0 & 3 & 2 & 2 \end{bmatrix}$$

The sum of the three largest entries in any row of A is 7 while the sum of the three largest entries in any column is 6.

Problem 1 was proposed by the author in the problem corner of Image, no. 17, 1996. (Image is a newsletter of the International Linear Algebra Society.) This article is based on the solutions sent by H J Werner, S W Drury, Agata Smoktunowicz and Alexander Kovacek which were published in Image, no. 18, 1997. The problem was also used in a selection test during IMOTC 1996.